Areal Co-ordinate Methods in Euclidean Geometry

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April 11, 2008

Introduction

In this article I aim to briefly develop the theory of areal (or 'barycentric') co-ordinate methods with a view to making them accessible to a reader as a means for solving problems in plane geometry. Areal co-ordinate methods are particularly useful and important for solving problems based upon a triangle, because, unlike cartesian co-ordinates, they exploit the natural symmetries of the triangle and many of its key points in a very beautiful and useful way.

Setting up the co-ordinate system

If we are going to solve a problem using areal co-ordinates, the first thing we must do is choose a triangle ABC, which we call the *triangle of reference*, and which plays a similar role to the axes in a cartesian co-ordinate system. Once this triangle is chosen, we can assign to each point P in the plane a unique triple (x, y, z) fixed such that $x + y + z = 1$, which we call the areal co-ordinates of P. The way these numbers are assigned can be thought of in three different ways, all of which are useful in different circumstances. I shall reserve the proofs that these three conditions are equivalent, along with a proof of the uniqueness of areal co-ordinate representation, for the appendix. The first definition we shall see is probably the most intuitive and most useful for working with. It also explains why they are known as 'areal' co-ordinates.

1st Definition: A point P internal to the triangle ABC has areal co-ordinates $\left(\frac{[PBC]}{[ABC]}, \frac{[PCA]}{[ABC]}, \frac{[PAB]}{[ABC]}\right)$. If a sign convention is adopted, such that a triangle whose vertices are labelled clockwise has negative area, this definition applies for all P in the plane.

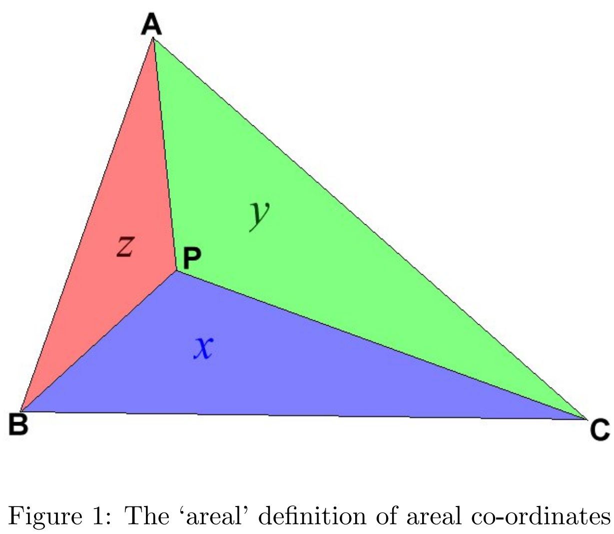

Figure 1: The 'areal' definition of areal co-ordinates

2nd Definition: If x, y, z are the masses we must place at the vertices A, B, C respectively such that the resulting system has centre of mass P, then (x, y, z) are the areal co-ordinates of P (hence the alternative name 'barycentric')

3rd Definition: If we take a system of vectors with arbitrary origin (not on the sides of triangle ABC) and let **a**,**b**,**c**,**p** be the position vectors of A, B, C, P respectively, then $\mathbf{p} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$ for some triple (x, y, z) such that $x + y + z = 1$. We define this triple as the areal co-ordinates of P.

There are some remarks immediately worth making:

- The vertices A, B, C of the triangle of reference have co-ordinates $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ respectively.
- All the co-ordinates of a point are positive if and only if the point lies within the triangle of reference, and if any of the co-ordinates are zero, the point lies on one of the sides (or extensions of the sides) of ABC.

The Equation of a Line

A line is a geometrical object such that any pair of nonparallel lines meet at one and only one point. We would therefore expect the equation of a line to be linear, such that any pair of simultaneous line equations, together with the condition $x+y+z = 1$, can be solved for a unique triple (x, y, z) corresponding to the areal co-ordinates of the point of intersection of the two lines. Indeed, it follows (using the equation $x+y+z = 1$ to eliminate any constant terms) that the general equation of a line is of the form

$$lx + my + nz = 0$$

where l, m, n are constants and not all zero. Clearly there exists a *unique* line (up to multiplication by a constant) containing any two given points $P(x_p, y_p, z_p), Q(x_q, y_q, z_q)$. This line can be written explicitly as

$$(y_p z_q - y_q z_p)x + (z_p x_q - z_q x_p)y + (x_p y_q - x_q y_p)z = 0$$

This equation is perhaps more neatly expressed in the determinant form (see Appendix 1):

$$\begin{vmatrix} x & x_p & x_q \\ y & y_p & y_q \\ z & z_p & z_q \end{vmatrix} = 0$$

While the above form is useful, it is often quicker to just spot the line automatically. For example try to spot the equation of the line BC, containing the points B(0,1,0) and C(0,0,1), without using the above equation.

Of particular interest (and simplicity) are *Cevian* lines, which pass through the vertices of the triangle of reference. We define a **Cevian through A** as a line whose equation is of the form $my = nz$. Clearly any line containing A must have this form, because setting $y = z = 0, x = 1$ any equation with a nonzero x coefficient would not vanish. It is easy to see that any point on this line therefore has form $(x, y, z) = (1 - mt - nt, nt, mt)$ where t is a parameter. In particular, it will intersect the side BC with equation $x = 0$ at the point $U(0, \frac{n}{m+n}, \frac{m}{m+n})$. Note that from definition 1 (or 3) of areal co-ordinates, this implies that the ratio $BU/UC = [ABU]/[AUC] = m/n$.

Example 1: Ceva's Theorem

We are now in a position to start using areal co-ordinates to prove useful theorems. In this section we shall state and prove (one direction of) an important result of Euclidean geometry known as Ceva's Theorem. The author recommends a keen reader only reads the statement of Ceva's theorem initially and tries to prove it for themselves using the ideas introduced above, before reading the proof given.

Ceva's Theorem: Let ABC be a triangle and let L, M, N be points on the sides BC, CA, AB respectively. Then the cevians AL,BM,CN are concurrent at a point P if and only if

$$\frac{BL}{LC} \times \frac{CM}{MA} \times \frac{AN}{NB} = 1$$

Partial proof: Suppose first that the cevians are concurrent at a point P, and let P have areal co-ordinates (p, q, r). Then AL has equation $qz = ry$ (following the discussion of Cevian lines above), so $L(0, \frac{q}{q+r}, \frac{r}{q+r})$, which implies $BL/LC = r/q$. Similarly, $CM/MA = p/r, AN/NB = q/p$. Taking their product we get $\frac{BL}{LC} \times \frac{CM}{MA} \times \frac{AN}{NB} = 1$, proving one direction of the theorem. I leave the converse to the reader.

The above proof was very typical of many areal co-ordinate proofs. We only had to go through the details for one of the three cevians, and then could say 'similarly' and obtain ratios for the other two by symmetry. This is one of the great advantages of the areal co-ordinate system in solving problems where such symmetries do exist (particularly problems symmetric in a triangle ABC: such that relabelling the triangle vertices would result in the same problem).

Areas and Parallel Lines

One might expect there to be an elegant formula for the area of a triangle in areal co-ordinates, given they are a system constructed on areas. Indeed, there is. If PQR is an arbitrary triangle with $P(x_p, y_p, z_p), Q(x_q, y_q, z_q), R(x_r, y_r, z_r)$ then

$$\frac{[PQR]}{[ABC]} = \begin{vmatrix} x_p & x_q & x_r \\ y_p & y_q & y_r \\ z_p & z_q & z_r \end{vmatrix}$$

An astute reader might notice that this seems like a plausible formula, because if P, Q, R are collinear, it tells us that the triangle PQR has area zero, by the line formula already mentioned. It should be noted that the area comes out as negative if the vertices PQR are labelled in the opposite direction to ABC.

It is now fairly obvious what the general equation for a line parallel to a given line passing through two points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ should be, because the area of the triangle formed by any point on such a line and these two points must be constant, having a constant base and constant height. Therefore this line has equation

$$\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ z & z_1 & z_2 \end{vmatrix} = k = k(x + y + z)$$

Where k is a real constant.

Exercise: (BMO1 2007/8 Q5) Given a triangle ABC and an arbitrary point P internal to it, let the line through P parallel to BC meet AC at M, and similarly let the lines through P parallel to CA,AB meet AB,BC at N,L respectively. Show that

$$\frac{BL}{LC} \times \frac{CM}{MA} \times \frac{AN}{NB} \leq \frac{1}{8}$$

To infinity and beyond

Before we start looking at some more definite specific useful tools (like the positions of various interesting points in the triangle), we round off the general theory with a device that, with practice, greatly simplifies areal manipulations. Until now we have been acting subject to the constraint that $x+y+z=1$. In reality, if we are just intersecting lines with lines or lines with conics, and not trying to calculate any ratios, it is legitimate to ignore this constraint and to just consider the points (x,y,z) and (kx,ky,kz) as being the same point for all $k \neq 0$. This is because areal co-ordinates are a special case of a more general class of co-ordinates called **projective homogeneous co-ordinates**[1], where here the projective line at infinity is taken to be the line $x+y+z=0$. This system only works if one makes all equations homogeneous (of the same degree in x,y,z), so, for example, $x+y=1$ and $x^2+y=z$ are not homogeneous, whereas $x+y-z=0$ and $a^2yz+b^2zx+c^2xy=0$ are homogeneous. We can therefore, once all our line and conic equations are happily in this form, no longer insist on $x+y+z=1$, meaning points like the incentre $(\frac{a}{a+b+c},\frac{b}{a+b+c},\frac{c}{a+b+c})$ can just be written (a,b,c), a significant advantage for the practical purposes of doing manipulations. However, if any ratios or areas are to be calculated, it is imperative that the co-ordinates are *normalised* again to make $x+y+z=1$. This process is easy: just apply the map

$$(x,y,z) \mapsto (\frac{x}{x+y+z},\frac{y}{x+y+z},\frac{z}{x+y+z})$$

Significant areal points and formulae in the triangle

We have seen that the vertices are given by $A(1,0,0), B(0,1,0), C(0,0,1)$, and the sides by $x=0, y=0, z=0$. In the section on the equation of a line we examined the equation of a cevian, and this theory can, together with other knowledge of the triangle, be used to give areal expressions for familiar points in Euclidean triangle geometry. We invite the reader to prove some of the facts below as exercises.

- Triangle centroid: $G(1,1,1)$. The midpoints of the sides BC, CA, AB are given by $(0,1,1),(1,0,1),(1,1,0)$ respectively.
- Centre of the inscribed circle: $I(a,b,c)$ (hint: use the angle bisector theorem)
- Centres of escribed circles: $I_a(-a,b,c), I_b(a,-b,c), I_c(a,b,-c)$
- Symmedian point: $K(a^2,b^2,c^2)$
- Centre of the circumcircle: $O(\sin 2A, \sin 2B, \sin 2C)$
- Orthocentre (meet of the altitudes): $H(\tan A, \tan B, \tan C)$

It should be noted that the rather nasty trigonometric forms of O and H mean that they should be approached using areals with caution, preferably only if the calculations will be relatively simple.

If the reader is familiar with isogonal and isotomic conjugation, it is interesting to find that the isogonal conjugate of a point (x,y,z) is $(a^2/x, b^2/y, c^2/z)$ (verify with G, K, I, I and O, H above), and the isotomic conjugate $(1/x, 1/y, 1/z)$.

Exercise: Let D, E be the feet of the altitudes from A and B respectively, and P, Q the meets of the angle bisectors AI,BI with BC,CA respectively. Show that D, I, E are collinear if and only if P, O, Q are.

[1] The author regrets that, in the interests of concision, he is unable to deal with these co-ordinates in this document, but strongly recommends Christopher Bradley's *The Algebra of Geometry*, published by Highperception, as a good modern reference also with a more detailed account of areals and a plethora of applications of the methods touched on in this document. Even better, though only for projectives and lacking in the wealth of fascinating modern examples, is E.A.Maxwell's *The methods of plane projective geometry based on the use of general homogeneous coordinates*, recommended to the present author by the author of the first book.

Distances and circles

We finally quickly outline some slightly more advanced theory, which is occasionally quite useful in some problems, We show how to manipulate conics (with an emphasis on circles) in areal co-ordinates, and how to find the distance between two points in areal co-ordinates. These are placed in the same section because the formulae look quite similar and the underlying theory is quite closely related. Derivations can be found in [1].

Firstly, the general equation of a conic in areal co-ordinates is, since a conic is a general equation of the second degree, and areals are a homogeneous system, given by

$$px^2 + qy^2 + rz^2 + 2dyz + 2ezx + 2fxy = 0$$

Since multiplication by a nonzero constant gives the same equation, we have five independent degrees of freedom, and so may choose the coefficients uniquely (up to multiplication by a constant) in such a way as to ensure five given points lie on such a conic.

In Euclidean geometry, the conic we most often have to work with is the circle. The most important circle in areal co-ordinates is the circumcircle of the reference triangle, which has the equation (with a, b, c equal to BC, CA, AB respectively)

$$a^2yz + b^2zx + c^2xy = 0$$

In fact, sharing two infinite points[2] with the above, a general circle is just a variation on this theme, being of the form

$$a^2yz + b^2zx + c^2xy + (x + y + z)(ux + vy + wz) = 0$$

We can, given three points, solve the above equation for u, v, w substituting in the three desired points to obtain the equation for the unique circle passing through them.

Now, the areal distance formula looks very similar to the circumcircle equation. If we have a pair of points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, which must be normalised, we may define the displacement PQ : $(x_2 - x_1, y_2 - y_1, z_2 - z_1) = (u, v, w)$, and it is this we shall measure the distance of. So the distance of a displacement $PQ(u, v, w), u + v + w = 0$ is given by

$$PQ^2 = -a^2vw - b^2wu - c^2uv$$

Since $u + v + w = 0$ this is, despite the negative signs, always positive unless $u = v = w = 0$.

Appendix 1: The Determinant of a 3×3 Matrix

Matrix determinants play an important role in areal co-ordinate methods. We define the **determinant** of a 3 by 3 square matrix A as

$$|A| = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix} = a_x(b_yc_z - b_zc_y) + a_y(b_zc_x - bx_cz) + a_z(b_xc_y - b_yc_x)$$

This can be thought of as (as the above equation suggests) multiplying each element of the first column by the determinants of 2x2 matrices formed in the 2nd and 3rd columns and the rows not containing the element of the first column. Alternatively, if you think of the matrix as wrapping around (so b_x is in some sense directly beneath b_z in the above matrix) you can simply take the sum of the products of diagonals running from top-left to bottom-right and subtract from it the sum of the products of diagonals running from bottom-left to top-right (so think of the above RHS as $(a_xb_yc_z + a_yb_zc_x + a_zb_xc_y) - (a_zb_yc_x + a_xb_zc_y + a_yb_xc_z)$). In any case, it is worth making sure you are able to quickly evaluate these determinants if you are to be successful with areal co-ordinates.

[2] All circles have two (imaginary) points in common on the line at infinity. It follows that if a conic is a circle, its behaviour at the line at infinity $x + y + z = 0$ must be the same as that of the circumcircle, hence the equation given.

Miscellaneous Exercises

Here we attach a selection of problems compiled by Tim Hennock, largely from UK IMO activities in 2007 and 2008. None of them are trivial, and some are quite difficult, with difficulty roughly proportional to number of asterisks.

1. (Pre-IMO training 2007) *
Let ABC be a triangle. Let D, E, F be the reflections of A, B, C in BC, AC, AB respectively. Show that D, E, F are collinear if and only if $OH = 2R$.

2. (Balkan MO 2005) **
Let ABC be an acute-angled triangle whose inscribed circle touches AB and AC at D and E respectively. Let X and Y be the points of intersection of the bisectors of the angles $\angle ACB$ and $\angle ABC$ with the line DE and let Z be the midpoint of BC. Prove that the triangle XYZ is equilateral if and only if $\angle A = 60°$

3. (NST 2007) **
Triangle ABC has circumcentre O and centroid M. The lines OM and AM are perpendicular. Let AM meet the circumcircle of ABC again at A'. Lines CA' and AB intersect at D and BA' and AC intersect at E. Prove that the circumcentre of triangle ADE lies on the circumcircle of ABC.

4. (IMO 2007) **
In triangle ABC the bisector of $\angle BCA$ intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

5. (RMM 2008) ***
Let ABC be an equilateral triangle. P is a variable point internal to the triangle, and its perpendicular distances to the sides are denoted by a^2, b^2 and c^2 for positive real numbers a, b and c. Find the locus of points P such that a, b and c can be the side lengths of a non-degenerate triangle.

6. (ISL 2006) ***
Let ABC be a triangle such that $\angle C < \angle A < \frac{\pi}{2}$. Let D be on AC such that $BD = BA$. The incircle of ABC touches AB at K and AC at L. Let J be the incentre of triangle BCD. Prove that KL bisects AJ.

7. (NST 2007) ***
The excircle of a triangle ABC touches the side AB and the extensions of the sides BC and CA at points M, N and P, respectively, and the other excircle touches the side AC and the extensions of the sides AB and BC at points S, Q and R, respectively. If X is the intersection point of the lines PN and RQ, and Y the intersection point of RS and MN, prove that the points X, A and Y are collinear.

8. (Sharygin GMO 2008) ***
Let ABC be a triangle and let the excircle opposite A be tangent to the side BC at A_1. N is the Nagel point of ABC, and P is the point on AA_1 such that $AP = NA_1$. Prove that P lies on the incircle of ABC.

9. (NST 2007) ****
Let ABC be a triangle with $\angle B \neq \angle C$. The incircle I of ABC touches the sides BC, CA, AB at the points D, E, F, respectively. Let AD intersect I at D and P.
Let Q be the intersection of the lines EF and the line passing through P and perpendicular to AD, and let X, Y be intersections of the line AQ and DE, DF, respectively. Show that the point A is the midpoint of XY.

10. (Sharygin GMO 2008) **
Given a triangle ABC. Point A_1 is chosen on the ray BA so that the segments BA_1 and BC are equal. Point A_2 is chosen on the ray CA so that the segments CA_2 and BC are equal. Points B_1, B_2 and C_1, C_2 are chosen similarly. Prove that the lines A_1A_2, B_1B_2 and C_1C_2 are parallel.

Acknowledgements

The author owes thanks to Christopher Bradley, Geoff Smith and David Loeffler, all of whom provided invaluable advice on the presentation of this article, and to Tim Hennock for typesetting the exercises.

References

The author suggests the following resources (particularly 1 and 2) for a more detailed and rigorous exposition of these modern methods, and would also invite an interested reader to visit reference 4 to get a feeling for how useful areal co-ordinates are in the classification of triangle centres, one of the great geometrical endeavours of recent times. The author is a frequent user of the AskNRICH forum, as are other competent users of areal co-ordinates, so any questions on the subject posted there should receive a reply.

[1] C.J. Bradley - Challenges in Geometry (OUP)
[2] C.J. Bradley - The Algebra of Geometry (Highperception)
[3] E.A. Maxwell - The methods of plane projective geometry based on the use of general homogeneous coordinates
[4] Clark Kimberling - Encyclopedia of Triangle Centers (http://faculty.evansville.edu/ck6/encyclopedia/ETC.html)