AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

2008 IMO Team Training

Exam T15

- Each question is worth 7 points.
- Time allowed is $4\frac{1}{2}$ hours.
- No books, notes or calculators permitted.
- Any questions must be submitted in writing within the first half hour of the exam.

1. Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be positive real numbers. Prove that

$$\frac{1}{x_1} + \frac{x_1}{x_2} + \frac{x_1x_2}{x_2} + \frac{x_1x_2x_3}{x_4} + \cdots + \frac{x_1x_2\ldots x_n}{x_{n+1}} \geq 4(1 - x_1x_2\ldots x_{n+1}).$$

2. Let X be a set of 10,000 integers, none of them being divisible by 47.

   Prove that there exists a 2007-element subset Y of X such that $a-b+c-d+e$ is not divisible by 47 for any $a, b, c, d, e \in Y$.

3. Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incentre. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F.

   Prove that points E, I and F are collinear.