2009 United Kingdom & Australia Pre-IMO Camp

Trinity College, Cambridge

2nd Test

Thursday 9 July

The Ashes

- Each question is worth 7 points.
- Time allowed is $4\frac{1}{2}$ hours.
- No books, notes or calculators permitted.

1. Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa,$$

then $a = b = c$.

2. Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$.

Show that the quadrilateral $ABCD$ is cyclic.

3. Let $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$ be a $(k+\ell)$-element set of real numbers contained in the interval $[0, 1]$; k and ℓ are positive integers. A k-element subset $A \subset S$ is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{\ell} \sum_{x_j \in S \setminus A} x_j \right| \leq \frac{k+\ell}{2k\ell}.$$

Prove that the number of nice subsets is at least

$$\frac{2}{k+\ell} \binom{k+\ell}{k}.$$