AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

2011 IMO Team Training

Exam T14

- Each question is worth 7 points.
- Time allowed is $4\frac{1}{2}$ hours.
- No books, notes or calculators permitted.
- Any questions must be submitted in writing within the first half hour of the exam.

2011 Mathematical Ashes

1. Let γ be a fixed circle in the plane and let P be a fixed point not on γ. Two variable lines ℓ and ℓ' through P intersect γ at points X and Y, and X' and Y', respectively. Let M and N be the points directly opposite P in circles PXX' and PYY', respectively.

   Prove that as the lines ℓ and ℓ' vary, there is a fixed point through which line MN always passes.

2. Find all pairs (m, n) of non-negative integers for which

$$m^2 + 2 \cdot 3^n = m(2^{n+1} - 1).$$

3. We are given a positive integer k and two other integers $b > w > 1$. There are two strings of pearls, a string of b black pearls and a string of w white pearls. The *length* of a string is the number of pearls on it. These strings are cut in steps by the following rules. In each step:

   (a) The strings are ordered by their lengths in a non-increasing order, that is later strings always have length not exceeding those of earlier strings. If there are some strings of equal lengths, then the white ones precede the black ones. Then the k first strings, if they consist of more than one pearl, are chosen; if there are less than k strings longer than 1, then all strings are chosen.

   (b) Next, each chosen string is cut into two parts differing in length by at most one.

   (For instance, if there are strings of 5, 4, 4, 2 black pearls, strings of 8, 4, 3 white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4)$, $(3, 2)$, $(2, 2)$ and $(2, 2)$, respectively.)

   The process stops immediately after the step when a first isolated white pearl appears.

   Prove that at this stage, there will still exist a string of at least two black pearls.