AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

2012 IMO Team Training

Exam T16

- Each question is worth 7 points.
- Time allowed is $4\frac{1}{2}$ hours.
- No books, notes or calculators permitted.
- Any questions must be submitted in writing within the first half hour of the exam.

2012 Mathematical Ashes

1. Let k be a given positive integer. A function $f\colon \mathbb{N}^+ \to \mathbb{N}^+$ satisfies

$$f(n) = \begin{cases} 1 & \text{if } n \le k+1 \\ f(f(n-1)) + f(n - f(n-1)) & \text{if } n > k+1. \end{cases}$$

   (a) Show that f is well posed, that is, show that $n - f(n-1)$ is always a positive integer.

   (b) Show that if m is any positive integer then $f^{-1}(m)$ is a finite non-empty set of consecutive positive integers.

   (Note: $\mathbb{N}^+$ denotes the set of positive integers and $f^{-1}(m)$ denotes the set of positive integers x such that $f(x) = m$.)

2. Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC, and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA, and AB.

   Prove that the midpoint of the segment EF lies on the line through the two intersection points of ω_E and ω_F.

3. Let m, n be given positive integers. Consider a rectangular $m \times n$ array consisting of mn unit squares. An integer is written in each unit square. A rectangle R consisting of one or more unit squares is called a *shelf* if there is an integer h such that

   (i) The number in each unit square in R is greater than h, and

   (ii) The number in each unit square not in R but sharing an edge or a point with the boundary of R is at most h.

   In addition, the entire rectangular array itself is considered to be a shelf.

   What is the maximum possible number of shelves?