AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

2014 IMO Team Training

Exam T16

- Each question is worth 7 points.
- Time allowed is $4\frac{1}{2}$ hours.
- No books, notes or calculators permitted.
- Any questions must be submitted in writing within the first half hour of the exam.

The 2014 Mathematical Ashes: AUS v UNK

1. Let D be the point on side BC such that AD bisects angle $\angle BAC$. Let E and F be the incentres of triangles ADC and ADB, respectively. Let ω be the circumcircle of triangle DEF. Let Q be the point of intersection of the lines BE and CF. Let H, J, K and M be the second points of intersection of ω with the lines CE, CF, BE and BF, respectively. Circles HQJ and KQM intersect at the two points Q and T.

   Prove that T lies on line AD.

2. Alison can perform the following operations on any finite simple[1] graph G:

   (a) If i is a vertex with odd degree in G, she can remove i and all edges involving i.

   (b) For each vertex $i \in G$, she creates a new vertex i'. Then she adds an edge between each pair i and i'. She also adds an edge between i' and j' iff there is an edge in G between i and j. No further edges are added or removed.

   Prove that, for any initial such graph, Alison may apply some sequence of these operations to generate a graph containing no edges.

3. Fix an integer $k \geq 2$. Two players, called Ana and Banana, play the following *game of numbers*: Initially, some integer $n \geq k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leq m' < m$ that is coprime to m. The first player who cannot move anymore loses.

   An integer $n \geq k$ is called *good* if Banana has a winning strategy when the initial number is n, and *bad* otherwise.

   Consider two integers $n, n' \geq k$ with the property that each prime number $p \leq k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

[1] *Finite* means a finite number of vertices. *Simple* means no loops (edges from i to i), and no multiple edges (two or more edges i to j).