AUS and UNK

2016 IMO Final Team Training

Exam F5

- Each question is worth 7 points
- Time allowed is $4\frac{1}{2}$ hours
- No books, notes or calculators permitted
- Any questions must be submitted in writing within the first half hour of the exam.

1. Let ABC be an acute triangle with orthocentre H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram ($AB \parallel GH$ and $BG \parallel AH$). Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$

2. Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{k a_k}{a_k^2 + k - 1} \quad \text{for every positive integer } k$$

Prove that

$$a_1 + a_2 + \cdots + a_n \geq n \quad \text{for every } n \geq 2.$$

3. Let S be a nonempty set of positive integers. We say that a positive integer is 'clean' if it has a unique representation as a sum of an odd number of distinct elements of S.

Prove that there exist infinitely many positive integers that are not clean.