AUS and UNK

2017 IMO Final Team Training

Exam F5

- Each question is worth 7 points.
- Time allowed is $4\frac{1}{2}$ hours.
- No books, notes or calculators permitted.
- Any questions must be submitted in writing within the first half hour of the exam.

1. Point A_1 lies inside acute scalene triangle ABC and satisfies

$$\angle A_1AB = \angle A_1BC \quad \text{and} \quad \angle A_1AC = \angle A_1CB.$$

Points B_1 and C_1 are similarly defined. Let G and H be the centroid and orthocentre, repsectively, of triangle ABC.

Prove that A_1, B_1, C_1, G, and H all lie on a common circle.

2. (a) Prove that for every positive integer n, there exists a fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b < \sqrt{n} + 1$ and $\sqrt{n} \le \frac{a}{b} \le \sqrt{n+1}$.

   (b) Prove there are infinitely many positive integers n such that there is no fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b < \sqrt{n}$ and $\sqrt{n} \le \frac{a}{b} \le \sqrt{n+1}$.

3. Let n be a given positive integer. Determine the smallest positive integer k with the following property:

> It is possible to mark k cells on a $2n \times 2n$ square array so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contains two marked cells.