Friday, July 6, 2018

Problem 1. Let $ABCD$ be a cyclic quadrilateral. Rays AD and BC meet at P. In the interior of the triangle DCP a point M is given, such that the line PM bisects $\angle CMD$. Line CM meets the circumcircle of triangle DMP again at Q. Line DM meets the circumcircle of triangle CMP again at R. The circumcircles of triangles APR and BPQ meet for a second time at S.

Prove that PS bisects $\angle BSA$.

Problem 2. Let $\mathbb{N}_0$ denote the set of non-negative integers. Find all functions $f\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f^{f(m)}(n) = n + 2f(m)$$

for all $m, n \in \mathbb{N}_0$ with $m \leq n$.

(Here f^k denotes the kth iterate of f.)

Problem 3. Let $n \geq 3$ be a given integer. For each convex n-gon $A_1A_2 \ldots A_n$ with no two sides parallel to each other, we construct a labelled graph G as follows.

- Its vertices $V_1, V_2, \ldots, V_n$ correspond to $A_1, A_2, \ldots, A_n$, respectively.
- An edge connects V_i to V_j whenever it is possible to draw two parallel lines, one passing through A_i and the other passing through A_j, such that apart from A_i and A_j, the whole n-gon lies strictly between these two lines.

Two such obtained graphs $G = V_1, V_2, \ldots, V_n$ and $H = W_1, W_2, \ldots, W_n$ are said to be the same if

V_iV_j is an edge of G if and only W_iW_j is an edge of H for all integers i, j with $1 \leq i < j \leq n$.

Find the number of different labelled graphs that can be obtained as described above, expressing your answer as a simple formula in terms of n.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points