Friday, September 18, 2020

Problem 1. The infinite sequence $a_0, a_1, a_2, \dots$ of (not necessarily different) integers has the following properties: $0 \le a_i \le i$ for all integers $i \ge 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \ge 0$.

Prove that all integers $N \ge 0$ occur in the sequence (that is, for all $N \ge 0$, there exists $i \ge 0$ with $a_i = N$.

Problem 2. We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \dots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S.

Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

Problem 3. Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \ne 2\cdot\angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$.

Prove that P lies on the diagonal CE if and only if

$$\text{area}(BCD) + \text{area}(ADE) = \text{area}(ABD) + \text{area}(ABP).$$