Tuesday, July 13, 2021

Problem 1. Given a positive integer k, show that there exists a prime p such that one can choose distinct integers $a_1, a_2, \ldots, a_{k+3} \in \{1, 2, \ldots p-1\}$ such that p divides $a_i a_{i+1} a_{i+2} a_{i+3} - i$ for all $i = 1, 2, \ldots k$.

Problem 2. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Given an integer $n \geq 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \ldots, n$ there exists some $x, y \in S$ such that $x - y = F_k$.

Problem 3. Let I and I_A be the integer and the A-excenter of an acute-angled triangle ABC with $AB < AC$. Let the incircle meet BC at D. The line AD meets BI_A and CI_A at E and F, respectively. Prove that the circumcircles of triangles AID and I_AEF are tangent to each other.

Language: English *Time: 4 hours and 30 minutes*
 Each problem is worth 7 points