Saturday, July 13, 2024

Problem 1. Let m and n be positive integers greater than 1. Consider an $m \times n$ grid with a coin lying tail-side up in each unit square of the grid. To perform a *move*, one must execute the following sequence of steps:

1. Select a 2×2 square S in the grid; then
2. Flip the coins in the top-left and bottom-right unit squares of S; then, finally
3. Flip the coin in either the top-right or bottom-left unit square of S.

Determine all pairs (m, n) for which it is possible that every coin shows head-side up after performing a finite number of moves.

Problem 2. Let $a_1 < a_2 < a_3 < \cdots$ be positive integers such that a_{k+1} divides $2(a_1 + a_2 + \cdots + a_k)$ for every positive integer k. Suppose that for infinitely many primes p, there exists a positive integer k such that p divides a_k.

Prove that for every positive integer n, there exists a positive integer k such that n divides a_k.

Problem 3. Let ABC be an acute-angled triangle with circumcircle ω. A circle Γ is internally tangent to ω at A and also tangent to BC at D. Let AB and AC intersect Γ at P and Q respectively. Let M and N be points on line BC such that B is the midpoint of DM and C is the midpoint of DN. Lines MP and NQ meet at K and intersect Γ again at I and J respectively. The ray KA meets the circumcircle of triangle IJK at $X \neq K$.

Prove that $\angle BXP = \angle CXQ$.

Language: English
Questions must not be published

Time: 4 hours and 30 minutes
Each problem is worth 7 points