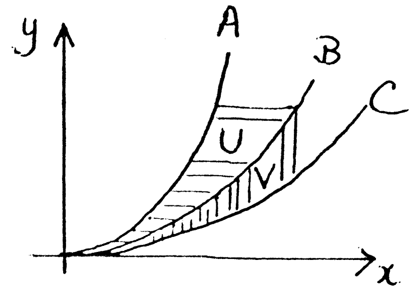
BRITISH MATHEMATICAL OLYMPIAD 1974

Time allowed: 3 hours

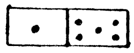
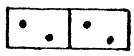
Each question should be answered on a fresh sheet of paper. Use one side of the paper only.

Do as much as you can; aim at answering whole questions. No tables, slide rules, calculators, or lists of formulae are allowed. Geometric instruments are allowed.

Please note the invigilator's instructions on a separate sheet.

A1. 

The curves A, B and C are related in such a way that B "bisects" the area between A and C, that is, the area of the region U is equal to the area of the region V at all points of the curve B. Find the equation of the curve B given that the equation of curve A is $y = \frac{4}{3}x^2$ and that the equation of curve C is $y = \frac{3}{8}x^2$.

B2. A domino can be represented by an unordered pair of integers. Thus  can be represented as (1,5) or (5,1) and the double  as (2,2). The set of all 15 dominoes containing two integers from 1, 2, 3, 4, 5 is partitioned into three subsets of five dominoes. The dominoes in each subset form a closed chain, that is (a,b)(b,c)(c,d)(d,e)(e,a) where a,b,c,d,e need not all be different. How many distinct partitions are there? (The order of the three subsets in the partition is immaterial.)

A3. Prove that it is impossible for all the faces of a convex polyhedron to be hexagons.

A4. M is a 16×16 matrix. Each element in the leading diagonal and each element in the bottom row (i.e. 16th row) is 1. Every other element of the matrix is $\frac{1}{2}$. Find the inverse of M.

A5. A bridge deal is defined as the distribution of 52 ordinary playing cards among four players so that each player has 13 cards. In a bridge deal what is the probability that just one player has a complete suit? (Leave your answer in factorials.)

B6. X and Y are the feet of the perpendiculars from P to CA and CB respectively, where P is in the plane of triangle ABC. PX = PY. The straight line through P which is perpendicular to AB cuts XY at Z. Prove that CZ bisects AB.

Turn over

B7. The roots of the equation $x^3 = bx + c$ ($bc \neq 0$, b and c real) are α, β and γ. Determine p, q and r in terms of b and c so that

$$\beta = p\alpha^2 + q\alpha + r, \quad \gamma = p\beta^2 + q\beta + r, \quad \alpha = p\gamma^2 + q\gamma + r$$

and state a condition which ensures that p, q and r are real.

A8. Let n be an odd prime number. It is required to write the product

$$\prod_{i=1}^{n-1} (x+i)$$

as a polynomial

$$\sum_{j=0}^{n-1} a_j x^j .$$

By considering the product $\prod_{i=1}^{n} (x+i)$ in two ways, establish the relations

$$\begin{aligned}
a_{n-1} &= 1, \\
a_{n-2} &= n(n-1)/2!\ , \\
2a_{n-3} &= n(n-1)(n-2)/3! + a_{n-2}(n-1)(n-2)/2!\ , \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(n-2)a_1 &= n + a_{n-2}(n-1) + a_{n-3}(n-2) + \ldots + 3a_2\ , \\
(n-1)a_0 &= 1 + a_{n-2} + \ldots + a_1\ .
\end{aligned}$$

Prove $n \mid a_j$ ($j = 1,2,\ldots,n-2$) and that $n \mid (a_0 + 1)$; and prove also that when x is an integer

$$n \mid (x+1)(x+2)\ \ldots\ (x+n-1) - x^{n-1} + 1\ .$$

Hence deduce Wilson's Theorem and Fermat's Theorem, namely, that when n is prime and x is not a multiple of n

(i) $\quad n \mid (n-1)! + 1$;

(ii) $\quad n \mid x^{n-1} - 1$.

($p \mid q$ means p divides q leaving no remainder.)

B9. A vertical uniform rod of length 2a is hinged at its lower end to a frictionless joint secured to a horizontal table. It falls from rest in this unstable position on to the table. Find the time occupied in falling. Comment on your answer.

[You may quote the result $\int(\operatorname{cosec} x)dx = \log\left|\tan \frac{1}{2}x\right|$ if you wish.]

B10. A right circular cone whose vertex is V and whose semi-vertical angle is α has height h and uniform density. All points of the cone whose distances from V are less than a or greater than b, where $0 < a < b < h$, are removed. A solid of mass M is left.

Given that the gravitational attraction that a point mass m at P exerts on unit mass at O is $(Gm/OP^3)\vec{OP}$, prove that the magnitude of the gravitational attraction of this solid on unit mass at V is

$$\tfrac{3}{2}GM(1 + \cos\alpha) \,/\, (a^2 + ab + b^2).$$