BRITISH MATHEMATICAL OLYMPIAD, 1975

24th March, 1975

Time allowed - 3 hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age and school on the top sheet of your answers. On each other sheet put your name and initials.

Do as much as you can. The earlier questions carry slightly fewer marks. Aim at answering whole questions. In any question, marks may be added for elegance and clarity or subtracted for obscure or poor presentation.

1. Given that x is a positive integer solve

$$[\sqrt[3]{1}] + [\sqrt[3]{2}] + \ldots + [\sqrt[3]{x^3 - 1}] = 400$$

(where $[z]$ means the integral part of z) and prove your solution is complete.

2. The first n prime numbers, $2,3,5,\ldots,p_n$ are partitioned into two disjoint sets A and B. The primes in A are $a_1,a_2,\ldots,a_h$ and the primes in B are $b_1,b_2,\ldots,b_k$ where $h+k=n$.

The two products $\prod_1^h a_i^{\alpha_i}$ and $\prod_1^k b_i^{\beta_i}$ are formed where the α_i and β_i are any positive integers.

If d divides the difference between these products prove that either $d = 1$ or $d > p_n$.

3. Use the pigeonhole principle (i.e. if more than n objects are put into n pigeonholes then at least one pigeonhole must contain more than one object) to answer the following question.

A disc S is defined as the set of all points P in a plane such that $|OP| \leqslant 1$, where $|OP|$ is the distance of P from O, a given point in the plane, called the centre of the disc.

Prove that if the disc S contains 7 points such that the distance from any of the 7 points to any other is greater than or equal to 1, then one of the 7 points is O.

Turn over

4. Three parallel lines AD, BE, CF are drawn through the vertices of triangle ABC meeting the opposite sides in D,E,F respectively.

The points P,Q,R divide AD, BE, CF respectively in the same ratio $k:1$ and P,Q,R are collinear. Find the value of k.

5. For any positive integer m you are given that

$$1 + \binom{2m}{1}\cos\theta + \binom{2m}{2}\cos 2\theta + \dots + \cos 2m\theta = (2\cos\tfrac{1}{2}\theta)^{2m}\cos m\theta$$

where there are $2m+1$ terms on the left hand side. Both these expressions are defined to be $f(\theta)$. The function $g(\theta)$ is defined by

$$g(\theta) = 1 + \binom{2m}{2}\cos 2\theta + \binom{2m}{4}\cos 4\theta + \dots + \cos 2m\theta .$$

Given that there is no rational k for which $\alpha = k\pi$ find the values of α for which

$$\lim_{m\to\infty} \frac{g(\alpha)}{f(\alpha)} = \frac{1}{2} .$$

6. Prove that if n is a positive integer greater than 1 and $x > y > 1$, then

$$\frac{x^{n+1} - 1}{x(x^{n-1} - 1)} > \frac{y^{n+1} - 1}{y(y^{n-1} - 1)} .$$

7. Prove that there is only one set of real numbers $x_1, x_2, \dots, x_n$ such that

$$(1 - x_1)^2 + (x_1 - x_2)^2 + \dots + (x_{n-1} - x_n)^2 + x_n^2 = \frac{1}{n+1} .$$

8. The interior of a wine glass is a right circular cone. The glass is half filled with water and then slowly tilted so that the water starts and continues to spill from a point P on the rim. What fraction of the whole conical interior is occupied by water when the horizontal plane of the water level bisects the generator of the cone furthest from P?