THE MATHEMATICAL ASSOCIATION

National Committee for Mathematical Contests

# British Mathematical Olympiad

10th March 1983

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order. Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. In the triangle ABC with circumcentre O, AB = AC, D is the midpoint of AB and E is the centroid of triangle ACD. Prove that OE is perpendicular to CD.

2. The Fibonacci sequence $\{f_n\}$ is defined by

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \qquad (n > 2).$$

Prove that there are unique integers $a, b, m$ such that $0 < a < m$, $0 < b < m$ and $f_n - anb^n$ is divisible by $m$ for all positive integers $n$.

3. The real numbers $x_1, x_2, x_3, \ldots$ are defined by

$$x_1 = a \neq -1 \quad \text{and} \quad x_{n+1} = x_n^2 + x_n \quad \text{for all } n \geq 1.$$

$S_n$ is the sum and $P_n$ is the product of the first $n$ terms of the sequence $y_1, y_2, y_3, \ldots,$ where $y_n = \frac{1}{1+x_n}$.

Prove that $aS_n + P_n = 1$ for all $n$.

4. The two cylindrical surfaces

$$x^2 + z^2 = a^2, \quad z > 0, \quad |y| \leq a$$

$$\text{and } y^2 + z^2 = a^2, \quad z > 0, \quad |x| \leq a$$

intersect and with the plane $z = 0$ enclose a dome-like shape which is here called a "cupola". The cupola is placed on top of a vertical tower of height $h$ whose horizontal cross-section is a square of side $2a$. Find the shortest distance from the highest point of the cupola to a corner of the base of the tower, over the surface of the cupola and tower.

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5. If 10 points are within a circle of diameter 5", prove that the distance between some 2 of the points is less than 2".

6. Consider the equation

$$\sqrt{(2p + 1 - x^2)} + \sqrt{(3x + p + 4)} = \sqrt{(x^2 + 9x + 3p + 9)} \qquad (1)$$

in which $x, p$ are real numbers and the square roots are to be <u>real</u> and <u>non-negative</u>. Show that if (1) holds then

$$(x^2 + x - p)(x^2 + 8x + 2p + 9) = 0.$$

Hence find the set of real numbers $p$ for which (1) is satisfied by exactly one real number $x$.