1. P, Q, R are arbitrary points on the sides BC, CA, AB respectively of triangle ABC. Prove that the triangle whose vertices are the centres of the circles AQR, BRP, CPQ is similar to triangle ABC.

2. Let \( a_n \) be the number of binomial coefficients \( \binom{n}{r} \) (\( 0 \leq r \leq n \)) which leave remainder 1 on division by 3 and let \( b_n \) be the number which leave remainder 2. Prove that \( a_n > b_n \) for all positive integers \( n \).

3. (i) Prove that, for all positive integers \( m \),

\[
(2 - \frac{1}{m})(2 - \frac{3}{m})(2 - \frac{5}{m}) \ldots (2 - \frac{2m-1}{m}) \leq m! .
\]

(ii) Prove that if \( a, b, c, d, e \) are positive real numbers then

\[
\left(\frac{a}{b}\right)^4 + \left(\frac{b}{c}\right)^4 + \left(\frac{c}{d}\right)^4 + \left(\frac{d}{e}\right)^4 + \left(\frac{e}{a}\right)^4 \\
\geq \frac{b}{a} + \frac{c}{b} + \frac{d}{c} + \frac{e}{d} + \frac{a}{e} .
\]
4. Let $N$ be a positive integer. Determine with proof the number of solutions of the equation

$$x^2 - [x^2] = (x - [x])^2$$

lying in the interval $1 \leq x \leq N$.

(For a real number $x$ the "integer part" $[x]$ is the largest integer which is $\leq x$.)

5. A plane cuts a right circular cone with vertex $V$ in an ellipse $E$ and meets the axis of the cone at $C$; $A$ is an extremity of the major axis of $E$. Prove that the area of the curved surface of the slant cone with $V$ as vertex and $E$ as base is

$$\frac{VA}{AC} \times \text{(area of } E\text{)}.$$

6. Let $a, m$ be positive integers. Prove that if there exists an integer $x$ such that $a^2x - a$ is divisible by $m$ then there exists an integer $y$ such that both $a^2y - a$ and $ay^2 - y$ are divisible by $m$.

7. ABCD is a quadrilateral which has an inscribed circle. With the side $AB$ is associated

$$u_{AB} = p_1 \sin \angle DAB + p_2 \sin \angle ABC$$

where $p_1$, $p_2$ are the perpendiculars from $A$, $B$ respectively to the opposite side $CD$. Define $u_{BC}$, $u_{CD}$, $u_{DA}$ likewise, using in each case perpendiculars to the opposite side. Show that

$$u_{AB} = u_{BC} = u_{CD} = u_{DA}.$$