NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

Tuesday 5 March, 1985

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. Two circles S_1 and S_2 each touch a straight line p at the same point P. All points of S_2 , except P, are in the interior of S_1 . A straight line q (i) is perpendicular to p; (ii) touches S_2 at R; (iii) cuts p at L; and (iv) cuts S_1 at N and M , where M is between L and R.

   (a) Prove that RP bisects angle MPN.

   (b) If MP bisects angle RPL, find, with proof, the ratio of the areas of S_1 and S_2 .

2. $a, b, c,$ are each numbers between 0 and 1. Prove that not all of $a(1-b)$, $b(1-c)$, $c(1-a)$ can be greater than $\frac{1}{4}$.

3. n and m are non-negative integers. Prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \ldots\ldots\ldots + (n+1-m)\binom{m}{m} = \binom{n+2}{m+2}$$

   where $\binom{r}{s}$ is the binomial coefficient $r(r-1)(r-2)\ldots(r-s+1)/s!$

4. The sequence f_n is defined by $f_0 = 1$, $f_1 = c$, where c is a positive integer, and for all $n > 1$,

$$f_n = 2f_{n-1} - f_{n-2} + 2 .$$

   Prove that for each $k \geqslant 0$ there exists h such that $f_k f_{k+1} = f_h$.

5. A cylindrical container has height 6 cm and radius 4 cm . It rests on a circular hoop which has also radius 4 cm and the hoop is fixed in a horizontal plane. The container rests with its axis horizontal and with each of its circular rims touching the hoop at two points. The cylinder is now moved so that each of its circular rims still touches the hoop at two points. Find, with proof, the locus of the centre of one of the cylinder's circular ends.

6. Show that the equation $x^2 + y^2 = z^5 + z$ has infinitely many solutions in positive integers x, y, z having no factor in common greater than 1.