NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

Friday 20th March 1987

Time allowed - 3½ hours

PLEASE READ THESE INSTRUCTIONS CAREFULLY:

Write on one side of the paper only. Use a fresh sheet or sheets of paper for each question. Arrange your answers in order. Put your full name, age (in years and months), home address and school on the top sheet of your answers. On each other sheet put your name and initials, and the number of the question.

There is no restriction on the number of questions which may be attempted. Nos 1-3 may be found easier than nos 4-7. Candidates who wish to be considered for the British I.M.O. team will be judged on their performance on the whole paper but additional weight will be given to the harder questions.

1. **(a)** Find, with proof, all integer solutions of

$$a^3 + b^3 = 9 .$$

(b) Find, with proof, all integer solutions of

$$35x^3 + 66x^2y + 42xy^2 + 9y^3 = 9 .$$

2. In a triangle ABC, $\angle BAC = 100°$ and $AB = AC$. A point D is chosen on the side AC so that $\angle ABD = \angle CBD$. Prove that $AD + DB = BC$.

3. Find, with proof, the value of the limit as $n \to \infty$ of

$$\sum_{r=0}^{n} \binom{2n}{2r} 2^r \Big/ \sum_{r=0}^{n-1} \binom{2n}{2r+1} 2^r .$$

Here $\binom{2n}{s}$ denotes a binomial coefficient.

4. /

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4. Let $P(x)$ be any polynomial with integer coefficients such that

$$P(21) = 17, \quad P(32) = -247, \quad P(37) = 33 .$$

Prove that if $P(N) = N + 51$ for some integer N, then $N = 26$.

5. A line parallel to the side BC of an acute-angled triangle ABC cuts the side AB at F and the side AC at E. Prove that the circles on BE and CF as diameters intersect on the altitude of the triangle drawn from A perpendicular to BC.

6. Find, with proof, the maximum value of

$$\frac{xyz}{(1+x)(x+y)(y+z)(z+16)}$$

for positive real numbers x, y, z.

7. Prove that if n and k are any positive integers then there exists a positive integer x such that $\frac{1}{2}x(x+1) - k$ is divisible by 2^n.