BRITISH MATHEMATICAL OLYMPIAD

Wednesday 16th January 1991

Time allowed – Three and a half hours

Instructions:  
- Start each question on a fresh sheet of paper.  
- Write on one side of the paper only.  
- On every sheet of working write the number of the question in the top left hand corner and your name, initials and school in the top right hand corner.  
- Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1, 2, 3, 4, 5, 6, 7 in order.  
- Staple all the pages neatly together in the top left hand corner.

1. Prove that the number  
   \[ 3^n + 2 \times 17^n \]
   where \( n \) is a non-negative integer, is never a perfect square.  
   [4 marks]

2. Find all positive integers \( k \) such that the polynomial \( x^{2k+1} + x + 1 \) is divisible by the polynomial \( x^k + x + 1 \).

   For each such \( k \) specify the integers \( n \) such that \( x^n + x + 1 \) is divisible by \( x^k + x + 1 \).  
   [5 marks]

3. \( ABCD \) is a quadrilateral inscribed in a circle of radius \( r \). The diagonals \( AC \), \( BD \) meet at \( E \).

   Prove that if \( AC \) is perpendicular to \( BD \) then
   \[
   EA^2 + EB^2 + EC^2 + ED^2 = 4r^2. 
   \]

   (*)

   Is it true that if (*) holds then \( AC \) is perpendicular to \( BD \)? Give a reason for your answer.  
   [6 marks]

Turn over ...
4. Find, with proof, the minimum value of \( (x + y)(y + z) \) where \( x, y, z \) are positive real numbers satisfying the condition
\[ xyz(x + y + z) = 1. \] [7 marks]

5. Find the number of permutations (arrangements)
\[ p_1, p_2, p_3, p_4, p_5, p_6 \]
of 1, 2, 3, 4, 5, 6 with the property:
For no integer \( n, 1 \leq n \leq 5 \), do \( p_1, p_2, \ldots, p_n \) form a permutation of 1, 2, \ldots, \( n \). [9 marks]

6. Show that if \( x \) and \( y \) are positive integers such that \( x^2 + y^2 - x \) is divisible by 2xy then \( x \) is a perfect square. [9 marks]

7. A ladder of length \( l \) rests against a vertical wall. Suppose that there is a rung on the ladder which has the same distance \( d \) from both the wall and the (horizontal) ground. Find explicitly, in terms of \( l \) and \( d \), the height \( h \) from the ground that the ladder reaches up the wall. [10 marks]