BRITISH MATHEMATICAL OLYMPIAD

Wednesday 16th January 1991

Time allowed – *Three and a half hours*

Instructions:

- *Start each question on a fresh sheet of paper.*
- *Write on one side of the paper only.*
- *On every sheet of working write the number of the question in the top left hand corner and your name, initials and school in the top right hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1, 2, 3, 4, 5, 6, 7 in order.*
- *Staple all the pages neatly together in the top left hand corner.*

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1. Prove that the number

$$3^n + 2 \times 17^n$$

where n is a non-negative integer, is never a perfect square.

[4 marks]

2. Find all positive integers k such that the polynomial $x^{2k+1} + x + 1$ is divisible by the polynomial $x^k + x + 1$.

For each such k specify the integers n such that $x^n + x + 1$ is divisible by $x^k + x + 1$.

[5 marks]

3. $ABCD$ is a quadrilateral inscribed in a circle of radius r. The diagonals AC, BD meet at E.

Prove that if AC is perpendicular to BD then

$$EA^2 + EB^2 + EC^2 + ED^2 = 4r^2. \qquad (*)$$

Is it true that if (*) holds then AC is perpendicular to BD? Give a reason for your answer.

[6 marks]

Turn over ...

4. Find, with proof, the minimum value of $(x + y)(y + z)$ where x, y, z are positive real numbers satisfying the condition

$$xyz(x + y + z) = 1.$$

[7 marks]

5. Find the number of permutations (arrangements)

$$p_1, p_2, p_3, p_4, p_5, p_6$$

of 1, 2, 3, 4, 5, 6 with the property:

For no integer n, $1 \le n \le 5$, do $p_1, p_2, \dots, p_n$ form a permutation of $1, 2, \dots n$.

[9 marks]

6. Show that if x and y are positive integers such that $x^2 + y^2 - x$ is divisible by $2xy$ then x is a perfect square.

[9 marks]

7. A ladder of length l rests against a vertical wall. Suppose that there is a rung on the ladder which has the same distance d from both the wall and the (horizontal) ground. Find *explicitly*, in terms of l and d, the height h from the ground that the ladder reaches up the wall.

[10 marks]