United Kingdom Mathematics Trust

United Kingdom Mathematics Trust
School of Mathematics, University of Leeds, Leeds LS2 9JT
tel 0113 343 2339 *email* enquiry@ukmt.org.uk
fax 0113 343 5500 *web* www.ukmt.org.uk

British Mathematical Olympiad Round 1 2020

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

The 2020 paper

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for is full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap? On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore important that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

In 2020 the BMO1, exceptionally, included a section where only answers were required. Partial credit was awarded for incorrect answers that indicated sensible engagement with the problems, but in general it was hard to score highly without correct answers. Candidates were also penalised for the inclusion of incorrect answers alongside correct ones, and in some cases careful checking would have led to significantly higher scorers.

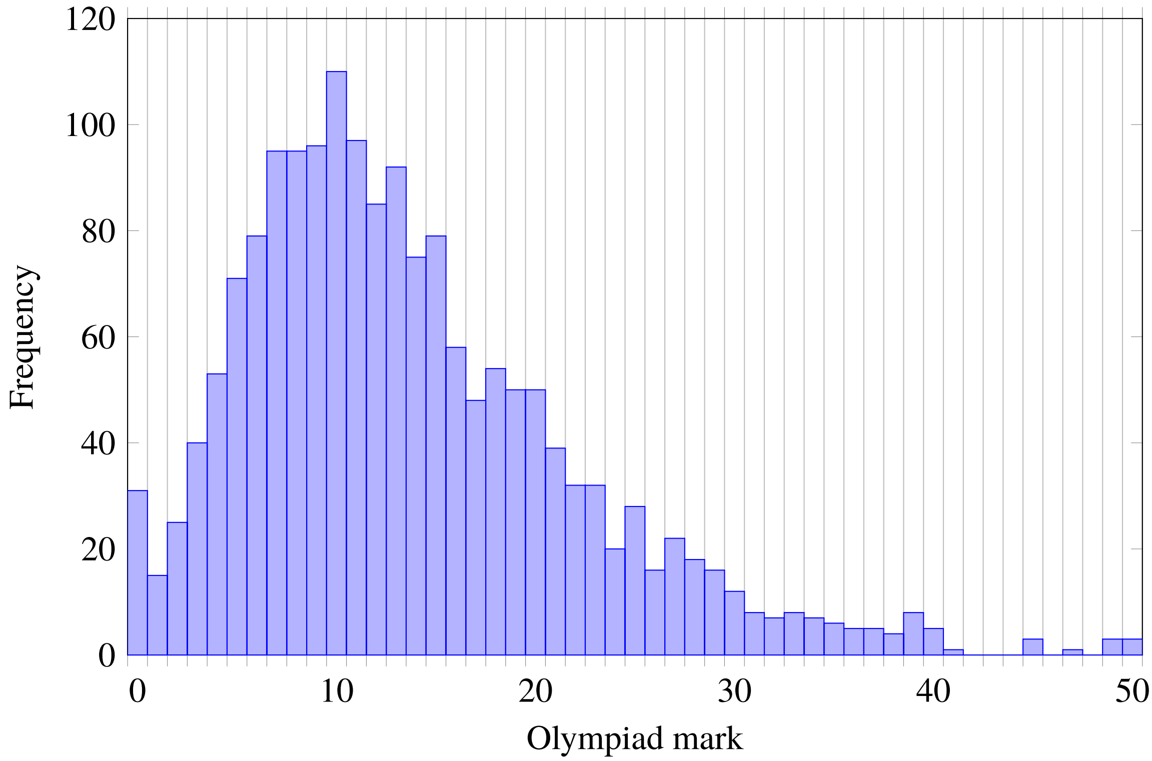
General comments

The format of the 2020 British Mathematical Olympiad Round 1 was a significant departure from previous years. This change was driven by factors relating to the Covid-19 pandemic, rather than any shift in ideology. There were concerns that schools would find administrating a 3.5 hour exam uniquely challenging this year, and also that the marking of the paper without the traditional on-site marking weekend might be problematic. The change in format addressed these issues, and the competition attracted a typical number of entries all of which were marked in a timely fashion. However, it seems unlikely that future BMOs will follow the 2020 format exactly.

It had been hoped that removing the need to 'write up' the first four questions, would give candidates more time to attempt section B. Certainly many candidates made good attempts at section A and at least one later question, but the answer only format also meant that arithmetical errors in easier questions were necessarily penalised more heavily than they might have been in a typical year. This, combined with the reduction in time, lead to a paper which candidates found particularly challenging. It was encouraging to see people rising to that challenge and, for the most part, engaging seriously with a number of the questions.

The 2020 British Mathematical Olympiad Round 1 attracted 1706 entries. The scripts were marked digitally from the 5th to the 11th of December by a team of Hugh Ainsley, Ann Ault, Eszter Backhausz, Jordan Baillie, Agnijo Banerjee, Sam Bealing, Emily Beatty, Jonathan Beckett, Phillip Beckett , Jamie Bell, Lex Betts, Robin Bhattacharyya, Tom Bowler, Magdalena Burrows, Shinwha Cha, Andrea Chlebikova, Arthur Conmy, James Cranch, Stephen Darby, Stefan Dixon, Ashling Dolan, Ceri Fiddes, Alison Fisher, Richard Freeland, James Gazet, Sarah Gleghorn, Amit Goyal, Daniel Griller, Peter Hall, Ben Handley, Stuart Haring, Adrian Hemery, Liam Hill, Michael Illing, Ian Jackson, Vesna Kadelburg, Jeremy King, Patricia King, David Knipe, Gerry Leversha, Warren Li, Sophie Maclean, Sam Maltby, Matei Mandache, David Mestel, Jordan Millar, Paul Murray, Joseph Myers, Michael Ng, Jenny Owladi, Eve Pound, Wendy Rathbone, Frankie Richards, Dominic Rowland, Paul Scarr, Amit Shah, Fiona Shen, Geoff Smith, Leona So, Anne Strong, Karthik Tadinada, Stephen Tate, Paul Walter, Zi Wang, Kasia Warburton and Dominic Yeo.

Mark distribution



The thresholds for qualification for BMO2 were as follows:

Year 13: 27 marks or more.

Year 12: 26 marks or more.

Year 11 or below: 24 marks or more.

Question 1

Alice and Bob take it in turns to write numbers on a blackboard. Alice starts by writing an integer a between -100 and 100 inclusive on the board. On each of Bob's turns he writes twice the number Alice wrote last. On each of Alice's subsequent turns she writes the number 45 less than the number Bob wrote last. At some point, the number a is written on the board for a second time. Find the possible values of a.

Proposed by Daniel Griller

Solution

There are four possible starting numbers a which generate a repeat: 0, 30, 42 and 45.

We call the terms of the sequence written on the board $a_1, a_2, a_3, \ldots$ where $a_1 = a$. We will consider a more general version of the problem by writing s in place of 45. Now the sequence is determined by the rules $a_{2n} = 2a_{2n-1}$ and $a_{2n+1} = a_{2n} - s$ for $n \geq 1$.

The sequence begins $a, 2a, 2a - s, 4a - 2s, 4a - 3s, 8a - 6s, 8a - 7s, \ldots$.

In general we see that $a_{2k} = 2^k a - (2^k - 2)s$ while $a_{2k+1} = 2^k a - (2^k - 1)s$. These can be proved formally by induction, but this is obviously not required for the question.

We now have two cases to consider:

Case 1: $a_{2k} = a$ for some $k \geq 0$.

In this case $(2^k - 1)a = (2^k - 2)s$. The numbers $2^k - 1$ and $2^k - 2$ cannot share any factors great than one since any factor of both would divide their difference. Thus we must have $2^k - 1$ is a factor of s.

Turning to the specific problem at hand, we need $2^k - 1$ to be a factor 45. The possible factors are $1 = 2^1 - 1, 3 = 2^2 - 1$ and $15 = 2^4 - 1$. These correspond to $a = 0, a = 30$ and $a = 42$ respectively, all of which work.

Case 2: $a_{2k+1} = a$ for some $k \geq 1$.

In this case we have $(2^k - 1)a = (2^k - 1)s$ and, since $k \neq 0$, this implies $a = s = 45$ which also works.

Alternative

It is also fairly straightforward to solve the problem by trying values of a in turn.

If at any stage $a_{2k} > 90$, then the subsequent odd and even numbered terms both form increasing sequences, so once their terms are larger than a, the number a can never be repeated. Similarly if $a_{2k} < 0$ at any stage, then the subsequent odd and even numbered terms both form decreasing sequences, so once their terms are less than a, the number a can never be repeated.

Most values of a in the range -100 to 100 give sequences which rapidly fall into the two categories above, meaning the four exceptional cases can be easily identified.

Question 2

A triangle has side lengths a, a and b. It has perimeter P and area A. Given that b and P are integers, and that P is numerically equal to A^2, find all possible pairs (a, b).

Proposed by Tom Bowler

Solution

We have that $P = 2a + b$ and, by Pythagoras' theorem, the height is $\sqrt{a^2 - \frac{b^2}{4}}$ which means that $A = \frac{b}{2}\sqrt{a^2 - \frac{b^2}{4}}$.

The condition in the question gives us the equation $2a + b = \left(\frac{b^2}{4}\right)\left(a^2 - \frac{b^2}{4}\right)$.

Thus $16(2a + b) = b^2(4a^2 - b^2) = b^2(2a - b)(2a + b)$.

We may divide by the (positive) $2a + b$ to get $b^2(2a - b) = 16$. (If $a = b = 0$ we do get a solution to the equation, but this corresponds to a single point rather than a triangle.)

Since b and $(2a - b) = P - 2b$ are both integers, b^2 must be a square factor of 16. The possible values are 1, 4 and 16 and we can substitute these into the equation in turn to find that $(a, b) = (\frac{17}{2}, 1), (3, 2)$ or $(\frac{5}{2}, 4)$.

Remark

Candidates were not penalised for including the solution $(a, b) = (0, 0)$ but it was not required to get full marks.

Remark

It is also possible to obtain the same equation linking a and b by using *Heron's formula* for the area of a triangle.

Question 3

A square piece of paper is folded in half along a line of symmetry. The resulting shape is then folded in half along a line of symmetry of the new shape. This process is repeated until n folds have been made, giving a sequence of $n + 1$ shapes. If we do not distinguish between congruent shapes, find the number of possible sequences when:

(a) $n = 3$;

(b) $n = 6$;

(c) $n = 9$.

(When $n = 1$ there are two possible sequences.)

Proposed by Daniel Griller

Solution

We classify the sequences of shapes according to the final shape in the sequence and make the following observations:

- The final shape can be an isosceles triangle if and only if the penultimate shape is either an isosceles triangle or a square.
- The final shape can be a square if and only if the penultimate shape is a rectangle in the ratio $1 : 2$.
- The final shape can be a rectangle in the ratio $1 : 2^k$ for $n \geq 1$ if and only if the penultimate shape is a rectangle in the ratio $1 : 2^{k-1}$ or $1 : 2^{k+1}$.

We now construct the Pascal-like triangle below.

n	$\triangle$	$\square$	1:2	1:4	1:8	1:16	1:32	1:64	1:128	1:256	1:512	Σ
0		1										1
1	1		1									2
2	1	1		1								3
3	2		2		1							5
4	2	2		3		1						8
5	4		5		4		1					14
6	4	5		9		5		1				24
7	9		14		14		6		1			44
8	9	14		28		20		7		1		79
9	23		42		48		27		8		1	149

Each entry is the number of n-fold sequences whose final shape is given by the column heading. The first two columns count sequences ending in isosceles triangles and squares respectively, the remaining columns count sequences ending in proper rectangles.

The observations above mean that:

- Each entry in the $\triangle$ column is the sum of entries in the $\triangle$ and $\square$ columns in the row above.
- Each entry in the $\square$ column is equal to the entry in the $1 : 2$ column in the row above.

- Each entry in the other columns is equal to the sum of the two entries 'diagonally above' it.

The answers can now be read off from the final column of the table, which records the sum of the entries in each row.

(a) 3 folds, 5 sequences; (b) 6 folds, 24 sequences; (c) folds, 149 sequences.

Alternative

There is a short cut: each number in the table contributes to two numbers in the row below, apart from the △ column which only contributes to one. Therefore each row sum is twice the previous row sum minus the △ number in the previous row. Thus we can work out all the relevant row sums by calculating with the much reduced table below.

n	△	□	1:2	1:4	1:8	Σ
0		1				1
1	1		1			2
2	1	1		1		3
3	2		2		1	5
4	2	2		3		8
5	4		5			14
6	4	5				24
7	9					44
8	9					79
9						149

Remark

There is a good deal more to explore in this question. For example, it turns out that the entries in the right hand portion of the table can be obtained by starting with a copy of Pascal's triangle and subtracting a second, slightly offset, copy of the same triangle. This can be verified by induction, but a direct counting argument is not so easy to come by.

One consequence of this observation is that the number of paths of length $2k$ which end in a square is given by $\binom{2k}{k} - \binom{2k}{k-1} = \frac{1}{k+1}\binom{2k}{k}$. These are the *Catalan numbers* which occur in a huge variety of different counting problems and certainly merit further reading.

If we call number of sequences ending in a triangle after either $2k$ or $2k - 1$ folds t_k, we obtain the sequence $1, 2, 4, 9, 23, \ldots$. The nth term is simply the sum of the first n Catalan numbers, but the sequence also satisfies the relation $t_{k+1} = \frac{1}{k+1}((5k-1)t_k - (4k-2)t_{k-1})$ which can be used to solve the original problem for much larger numbers of folds.

Question 4

In the equation

$$\mathsf{A}^{\mathsf{AA}} + \mathsf{AA} = \mathsf{B{,}BBC{,}DED{,}BEE{,}BBB{,}BBE}$$

the letters A, B, C, D and E represent different base 10 digits (so the right hand side is a sixteen digit number and AA is a two digit number). Given that C = 9, find A, B, D and E.

Proposed by Nick Mackinnon

Solution

First notice that $2^{22} = 2 \cdot 2^{21} = 2 \cdot 8^7 \leq 2 \cdot 10^7$ so $2 < A$.

Also $4^{44} = 2^{88} \geq 2^{80} = (2^{10})^8 \geq (10^3)^8 = 10^{24}$ so $A < 4$. Therefore $A = 3$.

We can now calculate the last two digits of $3^{33} + 33$ to find B and E. This can be done efficiently by repeatedly squaring to find the last two digits of $3^2, 3^4, 3^8, 3^{16}$ and 3^{32}.

The last two digits of 3^2 are 09.
The last two digits of 3^4 are 81.
The last two digits of $3^8 = 6561$ are 61.
The last two digits of 3^{16} are the same as the last two digits of $61^2 = 3721$, namely 21.
The last two digits of 3^{32} are the same as those of $21^2 = 441$, namely 41.

Multiplying by 3 and adding 33 shows that $BE = 56$.

Next we note that $3^{33} + 33$ is a multiple of 3, so the digit sum on the right hand side must also be a multiple of 3. Since $C = 9$ we may ignore it and the same applies to $E = 6$. There are exactly 9 Bs so these can also be ignored. We conclude that $2D$ must be a multiple of 3.

The digits $3, 6$ and 9 are already spoken for so $D = 0$.

Therefore $(A, B, D, E) = (3, 5, 0, 6)$.

Remark

The argument can be expressed more neatly using modular arithmetic, and there are a number of possible variations. For example, if we only work with the last digit we can still find $E = 6$ and hence $D = 0$. Next we can observe that $3^{33} + 33$ is two less than a multiple of 11. Now adding 2 to bot considering the alternating digit sum on the right gives enough information to determine B.

Markers' comments

There were a number of very good responses to the section A questions. However, some candidates lost marks through lack of adequate checking or unjustified pattern spotting. The first issue was most apparent in question 1 where many candidates listed one or more incorrect values of a alongside some or all of the correct ones. The nature of the question should make it natural to actually test each proposed value of a by writing out the sequence and waiting for the repeat, and candidates should be on the lookout for such opportunities to verify their solutions.

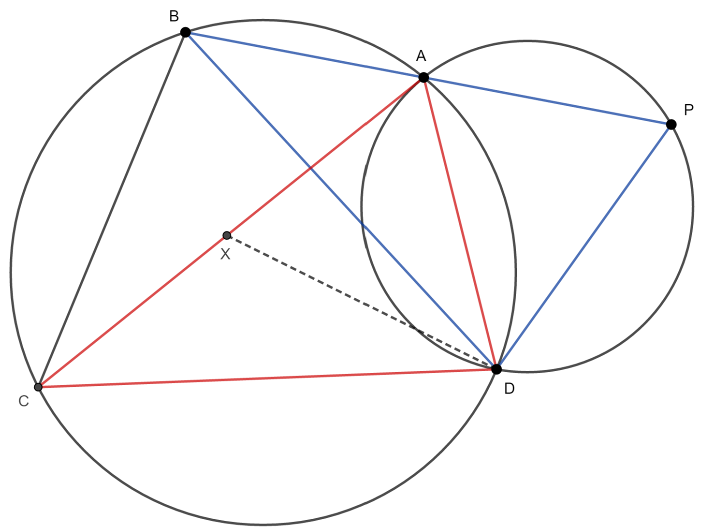
In question 3 a significant number of candidates gave the answers 5, 21, 89, which are the third, sixth and ninth Fibonacci numbers. The lesson here is that not every pattern in mathematics continues in the way one might initially expect. The sequence in this question begins 1, 2, 3, 5, 8, but the similarity to the Fibonacci numbers turns out to short-lived as the next term is not 13.

Question 5

Let points A, B and C lie on a circle Γ. Circle Δ is tangent to AC at A. It meets Γ again at D and the line AB again at P. The point A lies between points B and P. Prove that if $AD = DP$, then $BP = AC$.

Proposed by Dominic Yeo

SOLUTION



$AD = PD$ is given.

By the alternate segment theorem in Δ, we have $\angle CAD = \angle BPD$.
Then $\angle DCA = \angle DBA = \angle DBP$ by angles in the same segment in Γ.
These two angles show that triangles DCA and DBP are similar.

Then we have $AD = PD$, and so this similarity is a congruence.
So CAD is congruent to BPD (by 'Side-Angle-Side') and $BP = AC$.

ALTERNATIVE

Construct point X on ray AC with $AX = AP$.
Then $\angle DAX = \angle DPA = \angle PAD$ by the alternate segment theorem and then by ADP isosceles, so AD is perpendicular to PX as the angle bisector of isosceles PAX.
Therefore $PAXD$ is a kite, and $DX = PD = AD$.
This gives us $\angle DXC = 180^\circ - \angle AXD = 180^\circ - \angle DPA = 180^\circ - \angle PAD = \angle DAB$.
Then angles in the same segment gives $\angle ABD = \angle ACD$.
So we have DAB similar to DXC, and $AD = DX$ makes this a congruence.

So $XC = AB$ as required.

REMARK

It is also possible to solve the problem using the Sine Rule, or by quoting the fact that the unique spiral similarity sending segment BP to CA is centred at D.

Remark

Arguably the hardest part of this problem is constructing an accurate diagram. Adding points in the order in which they are specified in the question is unhelpful, and a better alternative is to begin by constructiong the circle Δ and the isosceles triangle ADP inside it.

Markers' comments

There were many excellent solutions to this problem. Most candidates found the simplest pair of congruent triangles. Some added a point X on AC such that $AX = AP$ and others added lines perpendicular to AP through D and perpendicular to AC through D. Both these methods needed extra work to find two pairs of congruent triangles but those who attempted them normally did so successfully. There were also some attempts using trigonometry, many of which were unsuccessful. A neat approach involving extending PD until it met circle Γ to create an isosceles trapezium was also successful.

Many of the solutions were very well explained. However, some candidates claimed equality of angles with no explanations and they were penalised for this. Although the standard GCSE theorems may be used without proof, it is important to make it clear at each step which circle theorem or which triangle is being used. In this question, the alternate segment theorem was crucial, (though knowledge of its name was not). Many did not know the theorem but managed to prove it from scratch, which did receive full credit but required extra work. However, those who claimed equality of pairs of angles without justification were penalised heavily.

In geometry, assuming special cases often reduces the problem to a simple one but does not solve the general case. Some assumed that PDC was a straight line or that PA was a diameter and got little credit. Another common error was claiming that a pair of triangles were congruent using two sides and one angle in the order 'Side-Side-Angle' which is not enough to prove congruence.

Question 6

Given that an integer n is the sum of two different powers of 2 and also the sum of two different Mersenne primes, prove that n is the sum of two different square numbers.

(A Mersenne prime *is a prime number which is one less than a power of two.)*

Proposed by Luke Pebody

Solution

We begin by writing

$$n = 2^a + 2^b - 2 = 2^c + 2^d$$

where $2^a - 1$ and $2^b - 1$ are Mersenne primes and (without loss of generality) we have $a > b$ and $c > d$.
Since $2^b - 1$ is prime, we must have $a > b \geq 2$, so $n = 2^a + 2^b - 2$ is divisible by 2 but not 4.
Since $n = 2^c + 2^d$ with $c > d$, we must have that $d = 1$ and $c \geq 2$.
Adding two to both sides of the original equation and dividing by 4 gives

$$2^{a-2} + 2^{b-2} = 2^{c-2} + 1.$$

The right-hand side is either 2 (if $c = 2$) or odd; in either case we must have $b = 2$.
Returning to the original equation we have $a = c$ and $n = 2^a + 2$.
Now a must be odd, otherwise $2^a - 1 = (2^{a/2} + 1)(2^{a/2} - 1)$, which is impossible since $2^a - 1$ is prime and $a \geq 3$. (Alternatively, if a were even, then $2^a - 1$ would be divisible by 3, which is impossible since it is prime and $a \geq 3$.)
To conclude, we write $a = 2k + 1$, so that the identity $n = 2^a + 2$ may be rewritten as

$$n = (2^k + 1)^2 + (2^k - 1)^2.$$

This exhibits n as the sum of two distinct squares.

Alternative

We begin by establishing that $d = 1$ as in the first solution. We then argue directly that c must be odd. For, if c were even, then $n = 2^c + 2$ would be divisible by 3. But all Mersenne primes other than 3 are one more than a multiple of 3, so there is no way to write a multiple of 3 as a sum of two distinct Mersenne primes.

We can then conclude as in the first solution since if $c = 2k + 1$, then $n = 2^{2k+1} + 2$ as before.

Alternative

Rewrite the equation as $2^a + 2^b = 2^c + 2^d + 2$. Since $2^b - 1$ is prime, $b \geq 2$ so the left hand side is even, which implies $d \geq 1$. However, the left hand side is a sum of exactly two distinct powers of 2, so by uniqueness of binary representation we have $d = 1$. This gives $2^a + 2^b = 2^c + 4$. The left hand side is not a power of 2, so again by uniqueness of binary representation, $b = 2$.

This shows that $a = c$. Since $2^a - 1$ is a Mersenne prime greater than three, and it is well known that Mersenne primes must be of the form $2^p - 1$ where p is prime, we see that a is odd and can conclude as before.

REMARK

The expression of $2^{2k+1} + 2$ as a sum of squares can be found using *Diophantus' identity* which states that $(a^2+b^2)(c^2+d^2) = (ab-cd)^2+(ad+bc)^2$. Setting $a = 2^k$ and $b = c = d = 1$ gives the result needed for the question. Candidates familiar with complex numbers may recognise the original identity, which is equivalent to the fact that the modulus function is multiplicative.

MARKERS' COMMENTS

This proved a popular question and it was good to see well over 100 complete solutions.

This problem can be broken down naturally into three distinct parts: finding the value b and d and showing $a = c$; showing that a is odd and finally finding an identity which demonstrates that n is the sum of two distinct squares. Candidates with a viable overall strategy could gain full or partial credit for each of these parts.

The most common approach to the first part was to divide both sides of the equation $2^a+2^b-2 = 2^c + 2^d$ by 2 and argue about parity. However, for this to be valid we must first check that b and d are non-zero. This is not difficult (we can observe that all Mersenne primes are odd and greater than 1), but failing to do it attracted a small penalty.

Arguments using binary representations as in the second alternative above were also common. Unfortunately many of these also failed to adequately consider small powers of two at the start.

Most candidates who who addressed the fact that a must be odd ruled out $a = 2k$ by considering $2^{2k} - 1 = (2^k - 1)(2^k + 1)$ and arguing that this is composite and so not a Mersenne prime. Some candidates overlooked the fact that the first bracket could be 1, yielding the Mersenne prime 3. Somewhat fortunately, 3 is the smaller of the Mersenne primes found in the first part and so a mark was not deducted for missing this case. The two other approaches in the alternative solutions above were also successfully employed by many candidates.

The third part was almost only ever successful when a candidate found the identity $2^a + 2 = 2^{2k+1} + 2 = (2^k + 1)^2 + (2^k - 1)^2$. Some candidates spotted a pattern by looking at small values of a and attempted to prove the required result by induction, this was rarely successful as many responses became convoluted and lost their train of thought. A very small number of candidates showed explicitly that $\frac{n}{2}$ was the sum of two distinct squares and stated that this implied that n was also the sum of two distinct squares.

Overall there were some excellent solutions to this quite technical problem but many candidates lost marks for not considering all relevant cases or for gaps in their reasoning. It is very important to remember that a mathematical proof should be communicated to the reader in a clear and concise manner but not so concise that the reader has to work hard to fill in gaps!

Question 7

Evie and Odette are playing a game. Three pebbles are placed on the number line; one at -2020, one at 2020, and one at n, where n is an integer between -2020 and 2020. They take it in turns moving either the leftmost or the rightmost pebble to an integer between the other two pebbles. The game ends when the pebbles occupy three consecutive integers.

Odette wins if their sum is odd; Evie wins if their sum is even. For how many values of n can Evie guarantee victory if:

(a) Odette goes first;

(b) Evie goes first?

Proposed by Daniel Griller

Solution

First note that the game must end after a finite number of moves, because the difference of the positions of the outer pebbles is a strictly decreasing sequence of positive integers until it is impossible to make a move (and so one player will win).

Claim 1: If Odette goes first she can force a win for any n.

Claim 2: If Evie goes first she can force a win if and only if n is odd.

These claims show the answer to the questions are 0 and 2020 respectively.

Proof of claim 1

Odette can play in such a way that after every turn Evie takes, the two outer pebbles are even.

If the configuration before Os turn is *even, even, even* O can move to another such configuration, or, if that is impossible, to a winning configuration for her.

If the configuration is *even, odd, even* and O has not already won, then she can move an outer pebble to an even number directly adjacent to the odd pebble. This will force E to move the odd pebble back between the two even ones.

This strategy ensures that the only way either player can end the game is by moving to consecutive numbers where the middle one is odd: a win for O.

Proof of claim 2

If the initial configuration is *even, even, even* before Es turn, then the argument from claim 1 shows that O can force a win.

If the initial configuration is *even, odd, even*, then E can move an outer even pebble to an odd number adjacent to the other even one. This forces O to move the even pebble, so the configuration after Os turn will have both outer pebbles odd.

Now the situation is identical to that one studied earlier but with the words even and odd interchanged, so E can arrange that in all subsequent configurations, the outer pebbles are both odd. The game must finish with adjacent places odd, even, odd and so Evie will win.

Markers' comments

This problem was found very hard. Only a minority of students found time to attempt it, and only a minority of attempts purported to be full solutions. A number of students had very good ideas, but wrote up only special cases: it is important to remember that a winning strategy has to explain what the winning player should do to win given any possible choice of moves by the other player.

In practice, those students frequently did well who observed (and who made clear that they observed) that moving two pebbles to adjacent positions forces the other player to move the outermost of the two on their next move.