UK Maths Trust

British Mathematical Olympiad

Round 1

Wednesday 15 November 2023

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Instructions

1. Time allowed: **$3\frac{1}{2}$ hours**.
2. **Full written solutions – not just answers – are required**, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.
3. **One complete solution will gain more credit than several unfinished attempts**. It is more important to complete a small number of questions than to try all the problems.
4. **Each question carries 10 marks**. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.
5. The use of rulers, set squares and compasses is allowed, but **calculators and protractors are forbidden**. You are strongly encouraged to use geometrical instruments to construct large, accurate diagrams for geometry problems.
6. Start each question on an official answer sheet on which there is a **QR code**.
7. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. **Please do not write your name or initials on additional sheets**.
8. **Write on one side of the paper only**. Make sure your writing and diagrams are clear and not too faint. (*Your work will be scanned for marking*.)
9. **Arrange your answer sheets in question order before they are collected**. If you are not submitting work for a particular problem, please remove the associated answer sheet.
10. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until **8am GMT on Friday 17 November** when the solutions video will be released at https://bmos.ukmt.org.uk
11. **Do not turn over until told to do so**.

Enquiries about the British Mathematical Olympiad should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. An unreliable typist can guarantee that when they try to type a word with different letters, every letter of the word will appear exactly once in what they type, and each letter will occur at most one letter late (though it may occur more than one letter early). Thus, when trying to type MATHS, the typist may type MATHS, MTAHS or TMASH, but not ATMSH.

Determine, with proof, the number of possible spellings of OLYMPIADS that might be typed.

2. The sequence of integers $a_0, a_1, \ldots$ has the property that for each $i \geq 2$, a_i is either $2a_{i-1} - a_{i-2}$ or $2a_{i-2} - a_{i-1}$.

Given that a_{2023} and a_{2024} are consecutive integers, prove that a_0 and a_1 are consecutive integers.

(Note that 6 *and* 7 *are consecutive integers, as are* 7 *and* 6.*)*

3. Let ABC be a triangle with $\angle ACB < \angle BAC < 90°$. Let X and Y be points on AC and the circle ABC respectively such that $X, Y \neq A$ and $BX = BY = BA$. Line XY intersects the circle ABC again at Z.

Prove that BZ is perpendicular to AC.

4. Find all positive integers n such that $n \times 2^n + 1$ is a square.

5. An artist arranges 1000 dots evenly around a circle, with each dot being either red or blue. A critic looks at the artwork and counts *faults*: each time two red dots are adjacent is one fault, and each time two blue dots are exactly two apart (that is, they have exactly one dot in between them) is another.

What is the smallest number of faults the critic could find?

6. For some integer $n > 4$ a convex polygon has vertices $v_1, v_2, \ldots, v_n$ in that cyclic order. All its edges are the same length. It also has the property that the lengths of the diagonals $v_1v_4, v_2v_5, \ldots, v_{n-3}v_n, v_{n-2}v_1, v_{n-1}v_2$ and v_nv_3 are all equal.

For which n is it necessarily the case that the polygon has equal angles?

