United Kingdom Mathematics Trust
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British Mathematical Olympiad Round 1 2024

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

The 2024 paper

General comments

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for are full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights. This means that many Olympiad mark schemes are such that a score of 5 out of 10 cannot be awarded. In general the logical structure of the mark scheme aims to reflect the logical structure of the problem while rewarding correct arguments more generously than correct calculations.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is often given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore vital that candidates taking Olympiad papers realise the importance of trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

The problems on the first half of this year's paper were a little more demanding than in 2023, leading to slightly lower average scores. However, it was still encouraging to see the vast majority of candidates making good progress on at least one problem. A fairly approachable geometry problem gave the strongest candidates time to engage with the final two problems, and an impressive number solved problem 5. Problem 6 proved hard enough to ensure that almost all candidates had something to think about throughout the exam.

Most candidates made a concerted effort to explain their reasoning and the effort put into writing up was appreciated by markers. It is worth noting that adding more mathematical notation does not always render an argument clearer. This was particularly apparent in problem 1. The easiest way to show that the relevant six numbers were *happy* was simply to write out the requisite circles of numbers. Many candidates who introduced algebra and tried to give more general constructions ended up losing marks by not explaining fully why the numbers in their constructions were, for example, all different.

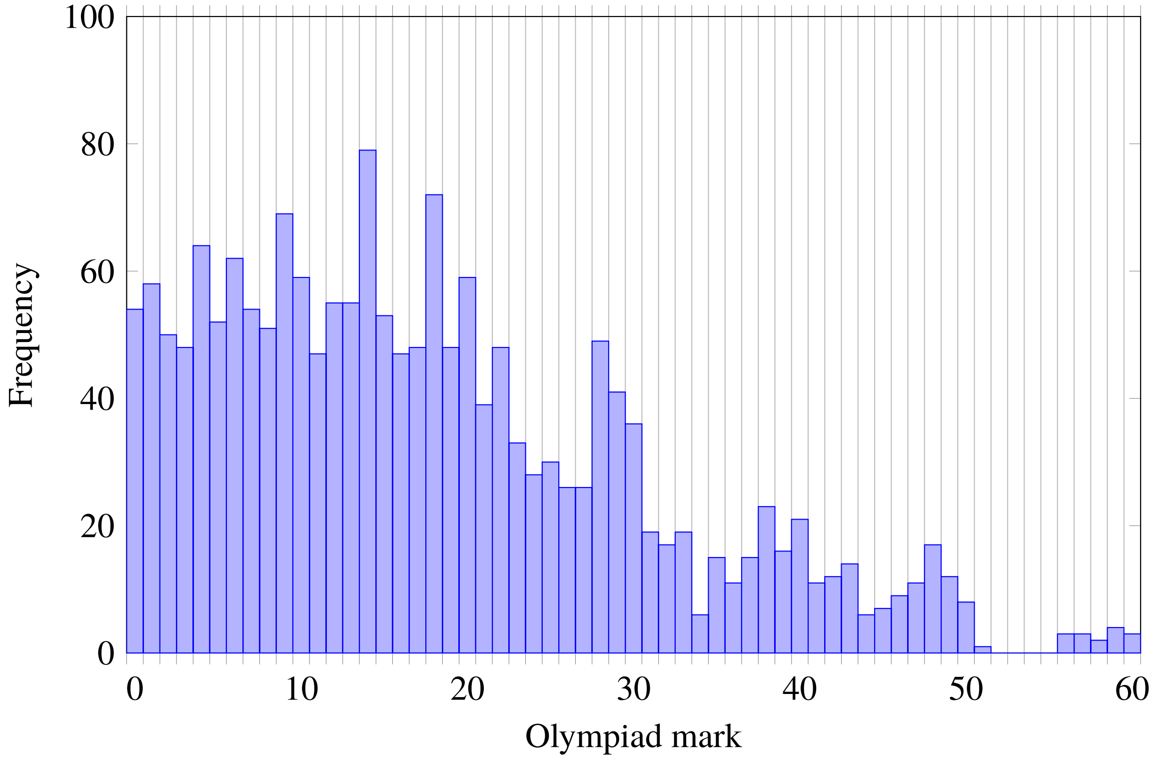
Problem 3 (as well as problem 1) provided the annual reminder of the importance of trying small examples. Candidates who considered what happens when the game is played starting with 2, then 3, then 4 and so on generally spotted the importance of parity and many went on to solve the problem. Those who dived in with an initial value of one million often missed the point and ended up attempting a delicate case analysis. This proved to be a much more difficult approach.

The 2024 British Mathematical Olympiad Round 1 attracted 1828 entries. The scripts were marked in London (with some remote markers) from the 6th to the 8th of December by a team of: Hugh Ainsley, Margaret Anthony, Eszter Backhausz*, Naomi Bazlov*, Sam Bealing*, Jonathan Beckett, Jamie Bell*, Robin Bhattacharyya, Tom Bowler*, Kaimyn Chapman, Raka Chattopadhyay, Andrea Chlebikova*, Volodymyr Chub, James Cranch, Juliette Culver, Stephen Darby, David Dyer , Paul Fannon, Chris Garton, Anthony Goncharov*, Amit Goyal, Aditya Gupta, Peter Hall, Ben Handley*, Stuart Haring*, Jon Hart, Alexander Hurst, Ian Jackson, Shavindra Jayasekera, Vesna Kadelburg, Thomas Kavanagh*, Patricia King, Isaac King, Jeremy King*, Hayden Lam, Sida Li, Elsa Lin, Thomas Lowe, Owen Mackenzie, Sam Maltby, Harry Metrebian, Kian Moshiri, Oliver Murray, Joseph Myers, Daniel Naylor, Huyen Ngoc Pham, Preeyan Parmar, Peter Price, Dominic Rowland, Heerpal Sahota, Geoff Smith*, Samuel Sturge, Rob Summerson, Stephen Tate, William Thomson, Tommy Walker Mackay*, Zi Wang, William Wu, Helen Xiaohui Chen, Lingde Yang, Dominic Yeo, Li Zhang, Haolin Zhao. (An asterisk shows that the marker was a problem captain.)

The problems were proposed by Geoff Smith, Geoff Smith, Sam Bealing, Dominic Yeo, Sam Cappleman-Lynes, and Jeremy King, respectively.

In addition to the written solutions in this report, video solutions can be found https://bmos.ukmt.org.uk/solutions/bmo1-2025/ and (with curated subtitles) at https://ukmt.org.uk/video-solutions-list

Mark distribution



The mean score was 17.8 and the median score was 16.

The thresholds for qualification for BMO2 were as follows:

Year 13: 43 marks or more.

Year 12: 40 marks or more.

Year 11 or below: 38 marks or more.

The thresholds for medals, Distinction and Merit were as follows:

Medal and book prize: 43 marks or more.

Distinction: 26 marks or more.

Merit: 10 marks or more.

Question 1

We say that the positive integer $n \geq 3$ is *happy* if it is possible to arrange n different positive integers in a circle such that two conditions are satisfied:

(a) If integers u and v in the circle are neighbours, then either u divides v or v divides u;

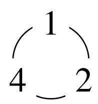
(b) If different integers u and v are in the circle but are not neighbours, then neither divides the other.

Determine, with proof, which positive integers n in the range $3 \leq n \leq 12$ are happy.

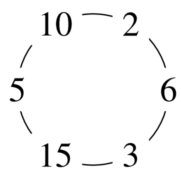
SOLUTION

The case $n = 3$ is unusual because every number is adjacent to every other, so condition (b) does not need to be considered. The diagram below shows one of many possible constructions showing 3 is happy.

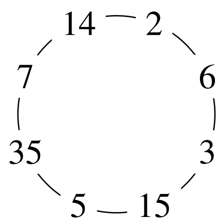
For even values of $n > 4$ we can show that n is happy by placing distinct primes in alternate spaces and placing the products of neighbouring primes between them as in the examples below. Checking that both conditions in the question are satisfied is straightforward. For $n = 4$ this construction does not work since if we use primes p_1 and p_2 both of the other numbers would equal p_1p_2 which is forbidden. This can be adapted by multiplying these two composite numbers by two different numbers neither of which divides the other, for example we can use $p_1^2p_2$ and $p_1p_2^2$ as shown below.



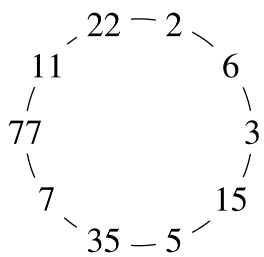
$n = 3$



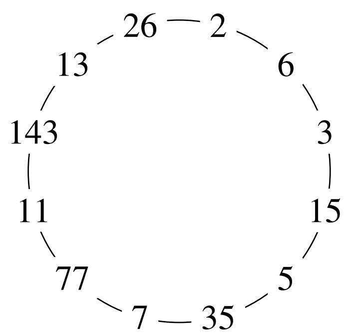
$n = 6$



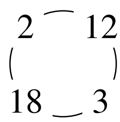
$n = 8$



$n = 10$



$n = 12$



$n = 4$

For the remaining cases, suppose that a, b and c are consecutive numbers round the circle. If $a < b < c$ then it must be the case that a divides b and that b divides c, but this would imply that a divides c which is impossible as a and c are not neighbours. Similarly $a > b > c$ would imply c divides a which is impossible. Thus if we place < and > signs between the numbers in the circle, they must alternate. However it is impossible for a odd number of these symbols to alternate round a circle. Thus if $n > 3$ is odd, n is not happy. So the required happy numbers are 3, 4, 6, 8, 10 and 12.

Remark

Solutions for the even n where $n = 4$ is not a special case can also be constructed; for example, place distinct primes in alternate spaces, and between them place products of neighbouring primes multiplied by yet another new prime. This gives $2, 30, 3, 42$ for the $n = 4$ case.

Remark

The question could easily have been phrased without the slightly unusual restriction to the numbers 3–12. This restriction was added to make it possible for candidates to solve the cases one at a time, and to emphasise the value of working through small cases systematically.

Markers' comments

Like many other Olympiad problems, there were two necessary ideas to prove: that certain numbers are happy (with a construction), and that the remaining numbers are not happy (with a proof that no construction is possible). Many candidates missed one of these points. Some proved that odds (other than 3) cannot be happy, and commented that the remaining numbers are therefore happy, implicitly using the erroneous reasoning that anything not yet proven unhappy must be happy. Others gave constructions for the happy numbers, usually involving primes, and then said that since this construction doesn't work for the remaining odds that they are therefore not happy. This is also erroneous, as other constructions might be possible.

The two special cases, of $n = 3$ and $n = 4$, also tripped up many candidates. This should be further evidence as to why we cannot use "my construction doesn't work for $n = 4$" as a proof that 4 is unhappy: in this case there is a slightly different construction that does work. Candidates who missed one or both of these cases could still score highly but not full marks.

Some common mistakes that candidates made included not checking their numerical answers, and overcomplicating the situation by introducing many variables. It can often be useful to consider specific numbers for small cases in mathematical questions rather than trying to be too general too soon.

Question 2

A magician performs a trick with a deck of n cards that are numbered from 1 to n. The magician prepares for the trick by putting the cards in an order of her choosing. Then she challenges a member of the audience to write an integer on a board. The magician turns over the cards one by one, in their pre-arranged order. Every time the magician turns over a card, the audience member multiplies the number on the board by -1, adds it to the number on the card, writes the result on the board, and erases the old number. The magician guarantees that, no matter which initial integer is chosen, the initial and final numbers will sum to 0.

Determine for which natural numbers n the magician can perform the trick. You must both prove that the trick is possible for the numbers you claim, and prove that it is not possible for any other numbers.

Solution

Suppose that the number written on the board at the start is x and that the magician turns over numbers $a_1, a_2, \ldots, a_n$ in that order. The sequence of numbers written on the board will be:

x
$a_1 - x$
$a_2 - (a_1 - x) = a_2 - a_1 + x$
$a_3 - (a_2 - a_1 + x) = a_3 - a_2 + a_1 - x$
$\ldots$
$a_n - a_{n-1} + a_{n-2} - \cdots + (-1)^n x$

When x is added to this final number the result must be zero. Since x can be any integer (including, for example, a very large one) it must be the case that $(-1)^n x + x = 0$. Thus n is odd.

We also need $a_n + a_{n-2} + a_{n-4} + \cdots = a_{n-1} + a_{n-3} + a_{n-5} + \ldots$. So the trick can be performed if and only if n is odd and it is possible to divide the numbers $1, 2, \ldots, n$ into two sets, of size $\frac{n-1}{2}$ and $\frac{n+1}{2}$ respectively, each of which has the same sum.

This common sum must be $\frac{1}{2}(1 + 2 + \cdots + n) = \frac{n(n+1)}{4}$. Since n is odd, this sum is only an integer if $n + 1$ is divisible by 4. So the trick cannot be performed unless $n = 4k + 3$ for some integer k.

We now show that if $n = 4k + 3$, then the trick can indeed be performed.

For $n = 3$ arranging the cards in the order 1, 3, 2 makes the final number $2 - 3 + 1 - x = -x$ as required.

For $n = 7$ the order 1, 3, 2, 4, 5, 7, 6 makes the final number $(6-7+5-4)+(2-3+1-x) = -x$.

In general if the trick can be made to work for some value of n, say $n = m$, it can also be made to work for $n = m + 4$. We simply add $m + 1, m + 2, m + 4, m + 3$ to the deck in that order. This will change the final number of the board by $((m + 3) - (m + 4) + (m + 2) - (m + 1)) = 0$.

Thus the trick can be performed for all $n = 4k + 3$ (by induction).

Markers' comments

There were many scripts which struggled to give a very clear reason why n must be odd for the trick to be possible, but the markers were generous on this point. There were several scripts where the case n even was eliminated, but then the possibility of n even continued to be analyzed in the rest of the argument!

A common gap was to state that: for odd n, if $1 + 2 + \cdots + n$ is even (or if $n = 4k + 3$), then the trick can be performed. This step is not obvious and requires justification.

If $1+2+\cdots+n$ is odd, then there is a short argument to show that the trick cannot be performed, but when is this sum odd? The key case is when $n = 4k + 1$ for some integer k. We did not accept "so $1 + 2 + \cdots + n$ is odd" because that could be pattern spotting (i.e. bluff). We required some sort of justification; this could be via the formula $n(n + 1)/2$ or by pointing out that there are an odd number of odd numbers in the range $1, 2, \ldots, n$, ideally with a little more explanation (for example by induction on k or by pairing off $2i + 1$ and $n - 2i$ for $0 \leq i < k$ with the extra term $2k + 1$). The markers accepted almost any evidence that the result was not pattern spotted.

Question 3

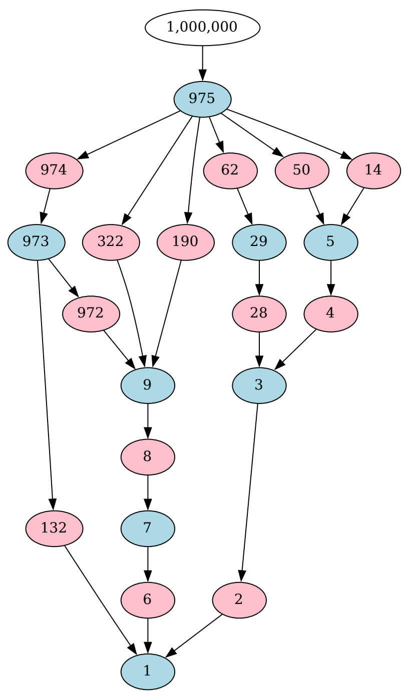
Rhian and Jack are playing a game in which initially the number 10^6 is written on a blackboard. If the current number on the board is n, a move consists of choosing two different positive integers a, b such that $n = ab$ and replacing n with $|a - b|$. Rhian starts, then the players make moves alternately. A player loses if they are unable to move.

Determine, with proof, which player has a winning strategy.

Solution

Rhian starts with 1 000 000 on the board, which is even an even number, so she has a winning strategy. She just needs to guarantee that she writes an odd number, which is always possible when she has an even on the board: $n = n \times 1$ and if n is even then $|n - 1|$ is odd. Jack, receiving an odd number, can only factorise it as a product of two odd numbers, as odd numbers do not have even factors. But the difference of two odd numbers is always even, so he must always write down an even number for Rhian.
Rhian can continue giving Jack odd numbers until she is finally able to give him 1, at which point she wins. It is important to note that 1 will indeed always be reached eventually, as if $ab = n$ then $1 \le a, b \le n$ so $|a - b| < n$ and so the value on the board decreases each move.

Alternative



Although it is difficult to do without the use of a computer, one might find specific numbers Rhian can pick that limit Jack's options, so it is possible to describe numerical strategies. The fastest such solution is represented on the tree above: the blue numbers are the choices Rhian should make and the pink ones list all the possible choices Jack has.

Markers' comments

Most successful candidates noticed that the game will always end on 1 as any other number n has at least two factors, 1 and n. Thus, working backwards from the end of the game, they observed that receiving 1 is a losing situation, 2 is winning because one can write 1, but then receiving 3 is also a losing position because one can only go to 2. Looking at a few more cases, they formed a conjecture that receiving an even number is a winning position whereas receiving an odd is losing.

To prove this, they had to include three parts in their solution: they had to show that even numbers can always be followed by an odd, that odd numbers must be followed by even, and that the numbers are decreasing so the game will end. Many candidates forgot to include the last part and received a small penalty for this omission.

Only a handful of students attempting a top-down solution, such as the alternative solution above, were successful in scoring more than one or two marks. That is because in order for such a solution to work, they had to list *all* possible ways Jack can react to Rhian's moves. Theoretically, it would be sufficient for Jack to have one good number to win, so even leaving out one possible option makes the solution incomplete.

Question 4

In the acute-angled triangle ABC we have $AB < AC < BC$. The midpoint of BC is M. There is a point P on the line segment AM such that $AB = CP$, and $\angle PAB = \angle BCP$.

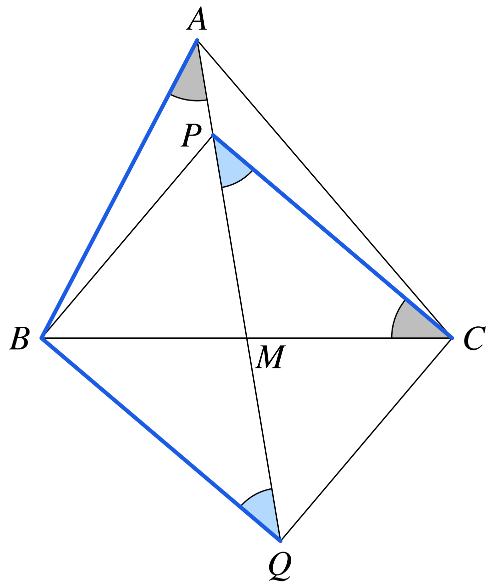
Prove that $\angle CPB = 90°$.

Solution

Most (non-trigonometric) solutions follow the following structure:

Step 1: Introduce a new point which helps you use the given conditions.
Step 2: Use the new point and the given conditions to find some equal angles.
Step 3: Use the equal angles to conclude that MB or MC is equal to MP.
Step 4: Conclude that M is the centre of the circumcircle of CPB, and so by angle in a semicircle, find that $\angle CPB = 90°$.

There are many variations within this broad structure. The one below was among the most common.

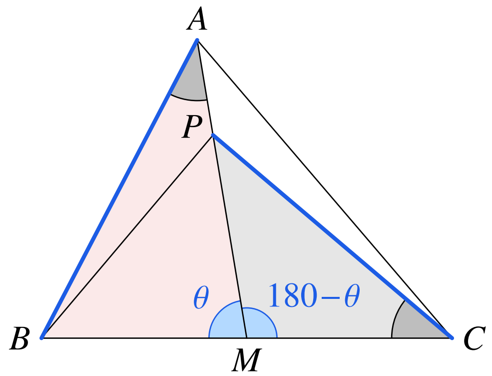


Step 1: Reflect P in the point M to get the point Q.
Step 2: Now the diagonals of $BQCP$ bisect each other so it is a parallelogram.
We have $PC = BQ = AB$, so ABQ is isosceles and $\angle QAB = BQA$.
Since $BQCP$ is a parallelogram, PC is parallel to BQ.
This means $\angle BQP = \angle CPQ = \angle MCP$.
Step 3: Now $\triangle MCP$ is isosceles, and we see that $MB = MP = MC$.
Step 4: This yields that M is the centre of the circle CPB with diameter BC and $\angle CPB = 90°$ since the angle in a semicircle is a right angle.

Remark

The fact that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram is well known. As such it can be quoted without proof in BMO1 if it is clearly stated. It is not to hard to prove since the diagonals split such a quadrilateral into pairs of triangles which are congruent by SAS.

Alternative



We use the Sine Rule on $\triangle MAB$ and $\triangle PMC$ to get the following equalities:

$$\frac{BM}{\sin \angle MAB} = \frac{AB}{\sin \angle BMA} = \frac{PC}{\sin \angle PMC} = \frac{PM}{\sin \angle MCP}$$

The first equality come from $\triangle MAB$, the last from $\triangle PMC$, the middle equality comes $AB = PC$ and $\angle BMA + \angle PMC = 180°$.
Now $\angle MAB = \angle MCP$, so $BM = PM$. So again, M is the centre of the the circumcircle of CPB with diameter BC and $\angle CPB = 90°$ from angle in a semicircle.

Alternative

There are a large number of possible ways to start this problem. Below is a list of some possible first steps which can be made to work. We hope some readers of this report will enjoy filling in the details for some of these.

- Define the point T on the line AM (extended beyond M) such that $AT = BC$.
- Construct X outside ABC such that the triangles XAB and CMP are congruent.
- Construct D between M and C such that $DC = AP$.
- Define Y to be the reflection of A in point M.
- Define Z be the intersection of lines AB and PC.
- Define W to be the intersection of the line AM (extended) and the circle ABC.
- Construct N between B and C such that $CN = AM$.
- Construct Q on AM such that triangles AQB and CMB are congruent.

Markers' comments

The majority of successful approaches to this question followed the format set out in the solution above. The key element was introducing an additional point to provide a more workable interpretation of the unusual conditions in the problem statement. The markers were impressed by the high level of creativity demonstrated, with more than ten distinct additional points introduced across various solutions.

Approaches of this nature lost marks where theorems (such as the "angle in a semicircle" theorem) were used implicitly rather than being clearly stated. Similarly, marks were deducted when a solution depended on the order of points along a line but failed to justify it. For example, if $Q \neq P$ is chosen on line AM such that $BP = BQ$, it must be established that Q lies strictly on segment AM rather than on the extension of ray AM beyond M for most solutions to work.

Another common method was using sine rule to handle the equal angles, as in the alternative solution above. When sine rule was used to show $BM = PM$, there were typically no issues. However, if the solution instead attempted to prove an angle equality, such as $\angle BPM = \angle MBP$, from equal sines ($\sin \angle BPM = \sin \angle MBP$), the argument needed to address both possible cases: either the angles are equal or they sum to 180°. Many students correctly ruled out the supplementary case by showing that the angles they were considering were part of a triangle. Those who did not provide such reasoning incurred a substantial penalty.

Question 5

Let p be a prime number, and let n be the smallest positive integer, strictly greater than 1, for which $n^6 - 1$ is divisible by p.

Prove that at least one of $(n+1)^6 - 1$ and $(n+2)^6 - 1$ is divisible by p.

Solution

We begin by noting that $n^6 - 1 = (n^3)^2 - 1$ is a difference of two squares so $n^6 - 1 = (n^3 - 1)(n^3 + 1)$. Next we observer that since $n^3 - 1$ has 1 as a root it has $(n-1)$ as a factor, while $n^3 + 1$ has $n + 1$ as a factor. Thus $n^6 - 1 = (n-1)(n^2 + n + 1)(n+1)(n^2 - n + 1)$.

Now suppose we have some prime p. If n is the least integer greater than 1 such that p divides $n^6 - 1$ (written $p \mid (n^6 - 1)$).

We have that $p \mid (n-1)(n^2 + n + 1)(n+1)(n^2 - n + 1)$ so p divides one of these four factors.

Moreover, provided $n > 3$, we know that p does not divide $(n-1)^6 - 1$ or $(n-2)^6 - 1$ since n is minimal. (We need $n > 3$ to ensure $n - 2 > 1$.)

Using the factorisation above and with some simplifications like $(n-1)^2 + (n-1) + 1 = n^2 - n + 1$ we note that:

p does not divide $(n-1)^6 - 1 = (n-2)(n^2 - n + 1)n(n^2 - 3n + 3)$;

p does not divide $(n-2)^6 - 1 = (n-3)(n^2 - 3n + 3)(n-1)(n^2 - 5n + 7)$.

These imply that, while p divides one of the four factors $(n-1)$, (n^2+n+1), $(n+1)$, (n^2-n+1) for any n where $p \mid (n^6 - 1)$, for the least $n > 1$ we must have either $p \mid (n^2 + n + 1)$ or $p \mid (n+1)$.

In the first of these case we see that p divides $(n+1)^6 - 1 = n(n^2 + 3n + 3)(n+2)(n^2 + n + 1)$; in the second case p divides $(n+2)^6 - 1 = (n+1)(n^2 + 5n + 7)(n+3)(n^2 + 3n + 3)$.

The problem is essentially solved except that we have assumed that $n > 3$. If $n = 2$, then $n^6 - 1 = 63 = 3^2 \times 7$, so $n = 2$ is the least n for $p = 3$ and $p = 7$. We have that 3 divides $4^6 - 1 = 4095$ and 7 divides $3^6 - 1 = 728$ so the problem conditions are satisfied for these primes. If $n = 3$ then $n^6 - 1 = 728 = 2^3 \times 7 \times 13$, so 3 is the minimal n for the primes 2 and 13. It is clear that 2 divides $5^6 - 1$ since it even. We also note that $4^6 - 1 = 4095 = 5 \times 819 = 5 \times 9 \times 91 = 3^2 \times 5 \times 7 \times 13$ so $13 \mid 4^6 - 1$ as required.

Alternative

It is possible to phrase a solution in the language of modular arithmetic which deals with the small cases slightly differently. As before we begin by observing that, for a given prime p, if n is the least integer greater than one such that $p \mid n^6 - 1$ then p divides one of $(n-1)$, $(n^2 + n + 1)$, $(n+1)$, $(n^2 - n + 1)$. We now consider these cases in turn.

Case I: If $p \mid n - 1$, then $n \equiv 1 \pmod{p}$.
Now $(n-2)^6 - 1 \equiv (-1)^6 - 1 \equiv 0 \pmod{p}$. Since n is minimal, it must be the case that $n - 2$ is not greater than 1, so $n = 3$ meaning $p = 2$. In this case we note that $2 \mid (n+2)^6 - 1$ as $5^6 - 1$ is even.

Case II: If $p \mid n^2+n+1$, then $n+1 \equiv -n^2 \pmod{p}$.
Now $(n+1)^6-1 \equiv (-n^2)^6-1 \equiv (n^6-1)(n^6+1) \equiv 0 \times 2 \pmod{p}$.

Case III: If $p \mid n+1$, then $n+1 \equiv 0 \pmod{p}$.
Now $(n+2)^6-1 \equiv 1^6-1 \equiv 0 \pmod{p}$.

Case IV: If $p \mid n^2-n+1$, then $n^2 \equiv -(n-1) \pmod{p}$.
In this case $(n-1)^6-1 \equiv (n^2)^6-1 \equiv (n^6-1)(n^6+1) \equiv 0 \times 2 \equiv 0 \pmod{p}$. Since n is minimal, it must be the case that $n-1$ is not greater than 1, so $n=2$ meaning that $p \mid 2^2-2+1$ so $p=3$. In this case we note that $3 \mid (2+2)^6-1$.

This completes the proof.

Markers' comments

This problem splits naturally into four separate cases depending on which of the factors of $n^6-1=(n-1)(n+1)(n^2+n+1)(n^2-n+1)$ is divisible by p. In order to solve the problem successfully, candidates had to identify that one of these factors was divisible by p and then handle all four of the resulting cases. Candidates who handled some but not all of the cases were generally awarded partial marks.

Some candidates partially factorised n^6-1, for example as $(n^3+1)(n^3-1)$, and attempted to handle some number of cases other than four. While some of these candidates were able to pick up partial credit for handling some of the cases, it is not feasible to solve this problem entirely without a complete factorisation. Candidates who did not factorise n^6-1 at all rarely obtained any substantial credit.

Some candidates correctly factorised n^6-1, but then falsely claimed that p had to be equal to one of the resulting factors. A counterexample to this claim is $p=19$, where $n=7$ and $p \mid n^2+n+1=57$. These candidates received very little credit, as all of their progress relied on an early incorrect step.

There were a few instances of candidates misunderstanding the conditions in the question. When the question states "let n be the smallest positive integer …", it means the smallest positive integer for the particular fixed value of p, not the smallest positive integer overall. Thus, it is an error to state that we must have $n=2$.

Moreover, we cannot make any assumptions about the prime p. Some candidates attempted to justify the (false) claim that one of the four factors of n^6-1 must be equal to p by stating that, e.g. if $p \mid n^2+n+1$, then the minimal possible value of n is $\frac{-1+\sqrt{4p-3}}{2}$ and therefore $p=n^2+n+1$. This argument is incorrect as it assumes that $\frac{-1+\sqrt{4p-3}}{2}$ is an integer, which is not true for most primes p.

Many (indeed, most) candidates who submitted otherwise successful solutions failed to consider exceptional cases that arise when $n=2$ or $n=3$. The reason why separate consideration of these cases is necessary is that arguments of the form "this contradicts n being minimal because I can find a smaller n that works" break down if the smaller value of n is too small to be allowed.

The case $n=2$ requires consideration of $p=3$ and $p=7$, the prime factors of $2^6-1=63$, while the case $n=3$ requires consideration of $p=2,7,13$, the prime factors of $3^6-1=728$.

However, some candidates were able to avoid handling all of these cases separately, as in the alternative solution above, by noting that the case $n = 2$ only causes an exception when $p \mid n^2 - n + 1$ and so $p = 3$, and the case $n = 3$ only causes an exception when $p = 2$.

Failing to handle the cases where n is small (or handling them incorrectly) was considered a minor omission, and candidates whose solutions were otherwise correct scored close to full marks, with a small number of marks deducted depending on the severity of the omission.

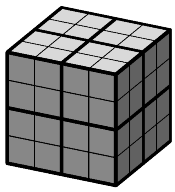
Question 6

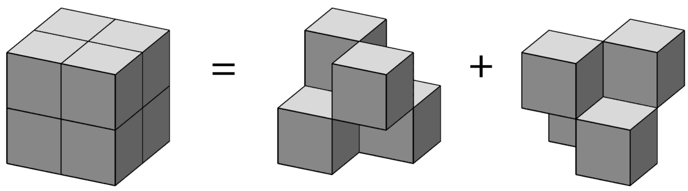
Björk has 64 sugar cubes, all of size $1 \times 1 \times 1$. Each sugar cube is either white or demerara or muscovado in flavour. She piles the sugar cubes into a neat $4 \times 4 \times 4$ cube.

Prove that there must be 12 sugar cubes of the same flavour which can be put into 6 disjoint pairs so that the distance between the centres of the cubes in each pair is the same.

Solution

The original cube can be split into eight disjoint (i.e. non-overlapping) subcubes of size $2 \times 2 \times 2$ as shown by the bold lines in the diagram on the left below. Each subcube can be split into two sets of four cubes whose centres form the vertices of a regular tetrahedron of edge length $\sqrt{2}$ as shown on the right below. By the pigeonhole principle, two of these vertices must have the same flavour, giving 16 same-flavour pairs at a distance of $\sqrt{2}$. By the extended pigeonhole principle, one of the flavours must appear as at least six of these pairs (since $16 > 5 \times 3$).





Alternative

We may work with regular tetrahedra of edge length $\sqrt{8} = 2\sqrt{2}$.

Markers' comments

Many contestants just constructed a single example with 6 same-flavour pairs at the same distance. The question was clear that there must be 6 pairs, however the flavours are distributed in the overall cube.

Many contestants correctly observed that there must be at least 22 cubes of the same flavour, by the extended pigeonhole principle. This was not rewarded, because it does not lead to any known solution. It is possible that having 22 cubes does *not* guarantee 6 pairs of that flavour at the same distance; the 6 pairs might come from one of the flavours with fewer cubes.

Next, they often placed the 231 distances between these 22 cubes into pigeonholes given by the 18 possible distances. This proves that at least 13 same-flavour pairs share the same distance. But converting this into six *disjoint* same-flavour pairs appears to be impossible.

Some contestants inserted these 22 cubes into the eight $2 \times 2 \times 2$ subcubes, or even the 16 regular tetrahedra. But they made assumptions about where these cubes had to go which were not guaranteed to be true.

Many contestants split the overall cube into sensible subproblems. But often these subproblems overlapped with each other, for example more than eight $2 \times 2 \times 2$ subcubes, or horizontal *and* vertical $4 \times 1 \times 1$ cuboids. These provided plenty of same-flavour pairs at the same distance, but they could not be shown to be disjoint.

The first step in this problem is to split the overall cube into *disjoint* subproblems. This ensures that any same-flavour pairs are disjoint. The second step is to control the distance between pairs. A distance of 1 is unfruitful; it is easy to construct an example with *no* same-flavour pairs at a distance of 1. As far as we know, $\sqrt{2}$ and $\sqrt{8}$ are the only options. The final step is to ensure that enough of these same-distance pairs have the same flavour overall.

A few contestants split the overall cube into eight $2 \times 2 \times 2$ subcubes. They claimed that each subcube provided 2 same-flavour pairs both at a distance of $\sqrt{2}$. With plenty of careful diagrams and explanation, they convinced the markers that they had dealt with all possible combinations of flavours in each subcube. This was not an easy task.

The majority of successful solutions explicitly or implicitly split each $2 \times 2 \times 2$ subcube into 2 sets of 4 cubes whose centres form the vertices of a regular tetrahedron of edge length $\sqrt{2}$. Occasionally they were penalised if their explanation of this splitting was unclear.