UK Maths Trust

British Mathematical Olympiad

Round 1

Wednesday 20 November 2024

supported by Jane Street Overleaf

Instructions

1. Time allowed: $\mathbf{3\frac{1}{2}}$ **hours**.
2. **Full written solutions – not just answers – are required**, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.
3. **One complete solution will gain more credit than several unfinished attempts**. It is more important to complete a small number of questions than to try all the problems.
4. **Each question carries 10 marks**. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.
5. The use of rulers, set squares and compasses is allowed, but **calculators and protractors are forbidden**. You are strongly encouraged to use geometrical instruments to construct large, accurate diagrams for geometry problems.
6. Start each question on an official answer sheet on which there is a **QR code**.
7. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. **Please do not write your name or initials on additional sheets**.
8. **Write on one side of the paper only**. Make sure your writing and diagrams are clear and not too faint. (*Your work will be scanned for marking*.)
9. **Arrange your answer sheets in question order before they are collected**. If you are not submitting work for a particular problem, please remove the associated answer sheet.
10. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until **12 noon GMT on Friday 22nd November** when the solutions video will be released at https://bmos.ukmt.org.uk and at ukmt.org.uk/competition-papers and also on *YouTube*. Candidates in time zones more than 5 hours ahead of GMT must sit the paper on Thursday 21st November (as defined locally). Do not share the content of the paper (including in email) until the videos have been published.
11. **Do not turn over until told to do so**.

Enquiries about the British Mathematical Olympiad should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. We say that the positive integer $n \geq 3$ is *happy* if it is possible to arrange n different positive integers in a circle such that two conditions are satisfied:

(a) If integers u and v in the circle are neighbours, then either u divides v or v divides u;

(b) If different integers u and v are in the circle but are not neighbours, then neither divides the other.

Determine, with proof, which positive integers n in the range $3 \leq n \leq 12$ are happy.

2. A magician performs a trick with a deck of n cards that are numbered from 1 to n. The magician prepares for the trick by putting the cards in an order of her choosing. Then she challenges a member of the audience to write an integer on a board. The magician turns over the cards one by one, in their pre-arranged order. Every time the magician turns over a card, the audience member multiplies the number on the board by -1, adds it to the number on the card, writes the result on the board, and erases the old number. The magician guarantees that, no matter which initial integer is chosen, the initial and final numbers will sum to 0.

Determine for which natural numbers n the magician can perform the trick. You must both prove that the trick is possible for the numbers you claim, and prove that it is not possible for any other numbers.

3. Rhian and Jack are playing a game in which initially the number 10^6 is written on a blackboard. If the current number on the board is n, a move consists of choosing two different positive integers a, b such that $n = ab$ and replacing n with $|a - b|$. Rhian starts, then the players make moves alternately. A player loses if they are unable to move.

Determine, with proof, which player has a winning strategy.

4. In the acute-angled triangle ABC we have $AB < AC < BC$. The midpoint of BC is M. There is a point P on the line segment AM such that $AB = CP$, and $\angle PAB = \angle BCP$.

Prove that $\angle CPB = 90°$.

5. Let p be a prime number, and let n be the smallest positive integer, strictly greater than 1, for which $n^6 - 1$ is divisible by p.

Prove that at least one of $(n + 1)^6 - 1$ and $(n + 2)^6 - 1$ is divisible by p.

6. Björk has 64 sugar cubes, all of size $1 \times 1 \times 1$. Each sugar cube is either white or demerara or muscovado in flavour. She piles the sugar cubes into a neat $4 \times 4 \times 4$ cube.

Prove that there must be 12 sugar cubes of the same flavour which can be put into 6 disjoint pairs so that the distance between the centres of the cubes in each pair is the same.

