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British Mathematical Olympiad Round 1 2025

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Olympiad Marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for are full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights. This means that many Olympiad mark schemes are such that a score of 5 out of 10 cannot be awarded. In general the logical structure of the mark scheme aims to reflect the logical structure of the problem while rewarding correct arguments more generously than correct calculations.

This approach is therefore rather different from what happens in public examinations such as GCSE and A-level, where credit is often given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore vital that candidates taking Olympiad papers realise the importance of trying to finish whole questions rather than attempting lots of disconnected parts.

The 2025 paper

General comments

This year's paper was somewhat more demanding than last year's with subtleties in questions 2 and 3, together with a demanding geometry problem, pushing the average scores down compared to 2024. Most candidates made meaningful progress on the first problem, though a significant minority misunderstood what was being asked. Reading mathematics is a skill that takes time to develop, but candidates are reminded that BMO1 problems are never trivial: if you believe you can fully resolve the question by observing that 2 cannot be written as a sum of three positive integers, it is likely you have not fully appreciated what is being asked.

In many cases candidates would have benefited from following the advice in the rubric to work in rough first then write up their best attempt. This would have made their ideas clearer, not only to the markers but also to the candidates themselves. It was common to see candidates missing particular cases on the first two problems: a little more organisation on the page might have made a big difference.

Problem 2 provided a firm reminder of the dangers of dividing by expressions which may be zero. Many candidates will likely be a little disappointed with low scores despite having all the correct solutions. The same could be said of problem 3, where many candidates made a crucial assumption about the arrangement of the pieces without adequate justification.

The second half of the paper was successful in keeping even the most able candidates busy. The handful scoring full marks have a great deal to be proud of.

As ever, there were a pleasing number of elegant and inventive solutions, and markers enjoyed some correct solutions they had not anticipated. Examples of such solutions are included as the final alternatives to problems 2 and 4 below.

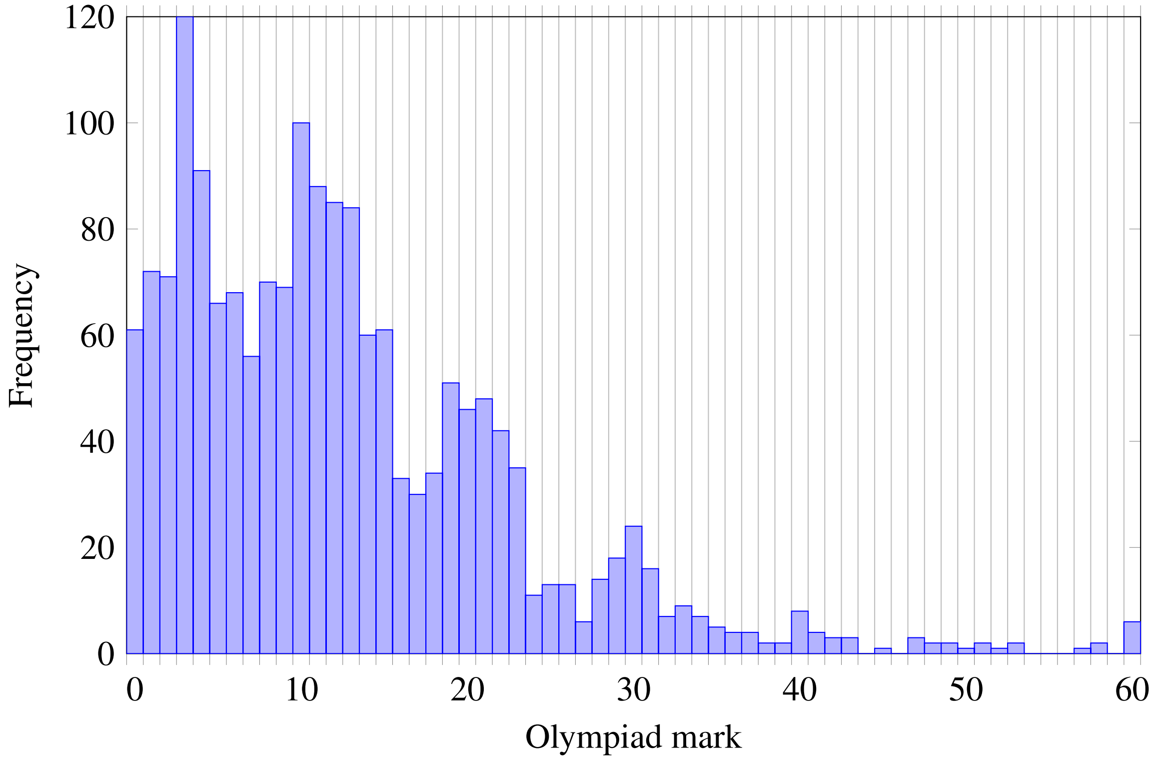
The 2025 British Mathematical Olympiad Round 1 attracted 1756 entries. The scripts were marked in London from the 5th to the 7th of December by a team of:

Eszter Backhausz*, Tessa Baker, Sam Bealing*, Jamie Bell*, Robin Bhattacharyya, Lucas Bowman, Abdellatif Charafi, Andrea Chlebikova, James Cranch, Gareth Davies, Chris Eagle, Thomas Frith, Carol Gainlall, Anthony Goncharov*, Daniel Gore, Amit Goyal, Zijie Guan, Ben Handley*, Stuart Haring, Jon Hart*, Ed Hart, Alexander Hurst*, Ian Jackson, Thomas Kavanagh, Jeremy King*, Patricia King, Isaac King, Hayden Lam, Larry Lau, Sida Li, Elsa Lin, Thomas Lowe, Owen Mackenzie, Georgina Majury, Sam Maltby*, Harry Metrebian, Oliver Murray, Joseph Myers, Jamie Nevill, Preeyan Parmar*, Huyen Ngoc Pham, Melissa Quail, Dominic Rowland, Heerpal Sahota, Gurjot Singh, Harcharan Singh Sidhu, Geoff Smith, Yang Song, Samuel Sturge, Jenni Voon, Tommy Walker Mackay*, Paul Walter, Zi Wang, Emma Wheeler, Lingde Yang*, Harvey Yau, Dominic Yeo, Siqin You, Li Zhang, Haolin Zhao. (An asterisk shows that the marker was a problem captain.)

The problems were proposed by Richard Freeland, Sam Cappleman-Lynes, Richard Freeland, Gerry Leversha, Dominic Rowland and Dominic Rowland, respectively.

In addition to the written solutions in this report, video solutions can be found at https://bmos.ukmt.org.uk/solutions/bmo1-2026/ and (with curated subtitles) at https://ukmt.org.uk/video-solutions-list

Mark distribution



The mean score was 12.6 and the median score was 11.

The thresholds for qualification for BMO2 were as follows:

Year 13: 31 marks or more.

Year 12: 30 marks or more.

Year 11 or below: 27 marks or more.

The thresholds for medals, Distinction and Merit were as follows:

Medal and book prize: 31 marks or more.

Distinction: 18 marks or more.

Merit: 7 marks or more.

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Question 1

A *ramp* is a sequence of three different positive integers a, b, c such that a is a factor of b and b is a factor of c. For every prime number p and every positive integer n, determine with proof whether p^n can be expressed as the sum of a ramp.

Solution

We claim that the only prime powers that are not the sum of a ramp are 2, 3, 4, 5 and 8.

If p is odd and $p^n \geq 7$, then if we let $a = 1$, $b = 2$ and $c = p^n - 3$, we see that p^n is the sum of a ramp.

We note that $16 = 1 + 3 + 12 = 1 + 5 + 10$ is a sum of a ramp (in two different ways) and observe that if a, b, c is a ramp then ka, kb, kc is also a ramp for any positive integer k. This shows that 2^n is the sum of a ramp for any $n \geq 4$.

The smallest sum of three distinct integers is 6, so none of 2, 3, 4 and 5 is the sum of a ramp.

This leaves 8, which can be written as the sum of distinct integers in two ways: $8 = 1 + 2 + 5 = 1 + 3 + 4$. Neither of these is a ramp.

Remark

Many candidates correctly rephrased the problem by observing that if a, b, c is a ramp then $b = ad$ and $c = be = ade$ where d and e are integers bigger than 1, so the sum of the ramp is $a(1 + d(1 + e))$.

Remark

It is not too difficult to classify all integers that are sums of ramps. The argument above already shows that any integer with an odd factor greater than 5, or a factor of 16 is the sum of a ramp.

This only leaves integers of the form $2^x 3^y 5^z$ where $x \leq 3$ and $y + z \leq 1$.

We note that $10 = 1 + 3 + 6$ so 10, 20 and 40 are sums of ramps. It remains to check 12 and 24.

Suppose, for contradiction, that $24 = a(1 + d(1 + e))$. We know $d, e > 1$ so $1 + d(1 + e) \geq 7$ so $a < 4$ but if $a = 3$ the $1 + d(1 + e) = 8$ and 8 is not the sum of a ramp. Thus $a = 1$ or 2, which means $1 + d(1 + e)$ is a multiple of 6. This in turn means that $d(1 + e)$ is not divisible by 2 or 3 so d and $1 + e$ are odd numbers different from 3. However this shows $d(1 + e) \geq 5 \times 5$ which is impossible. We have shown that 24 is not sum of a ramp, so 12 cannot be either.

We may summarise this as follows: the only integers that are not sums of ramps are 5 and the factors of 24.

Markers' comments

This problem was attempted by most candidates. Many scored highly by showing ramps summing to p^n exist for all but a short finite list of prime powers. To get full marks, the candidate also had to show that ramps did not exist with sums equal to 2, 3, 4, 5, and 8.

There were several reasons why candidates did not score well. Neglecting the case $p = 2$ was common, as was assuming that $n > 1$. Both errors lead to an infinite number of prime powers not being considered. Another common mistake was to describe a triple of numbers algebraically and claim it was a ramp without adequately checking that the numbers were in fact all different. Sometimes this resulted in a claim that, for instance, 5^1 could be expressed as the sum of the 'ramp' $(1, 2, 5^1 - 3)$, but failure to engage with the 'different numbers' condition was penalised even when candidates did not explicitly claim an incorrect case.

A small number of candidates demonstrated how some ramps could be constructed and then claimed that anything that did not fall into that construction could not be the sum of a ramp. For example, having seen that $(1, 2, p^n - 3)$ worked for odd p and $(1, 3, 2^n - 4)$ worked for even n, some said 2^5 could not be the sum of a ramp.

A surprising number of candidates misinterpreted the problem, thinking that they were being asked to determine whether the statement 'there is a ramp for every p and n' was true. This statement can be proved false simply by noting that the smallest ramp sums to 7, so, for example, 2^1 is not the sum of a ramp. Misreading the question this way initially is understandable, but the rubric makes it clear that each question carries 10 marks. Candidates who produce extremely short solutions are advised to ask themselves whether their work could plausibly be worth 10 marks before moving on to the next problem.

Question 2

Find all triples (x, y, z) of real numbers satisfying the equations

$$x^2 + 2yz = 4, \quad y^2 + 2zx = 4, \quad z^2 + 2xy = 1.$$

SOLUTION

Adding all three equations we obtain $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x + y + z)^2 = 9$ and so $x + y + z = \pm 3$.

Subtracting the first two equations is an attractive idea because the right-hand side will be 0, and the left-hand side will factorize. We obtain

$$(x - y)(x + y) - 2z(x - y) = (x - y)(x + y - 2z).$$

Therefore $x = y$ or $x + y = 2z$.

(We can also obtain this result by equating the first two equations, adding z^2 to both sides and rearranging to give $(x - z)^2 = (y - z)^2$ so $x - z = \pm(y - z)$.)

First we consider the case that $x = y$. In this situation, subtract the second equation from the third: $z^2 + 2x^2 - x^2 - 2zx = -3$ which factorises to give $(z - x)^2 = -3$. Squares are not negative so we can deduce that $x \neq y$ and so $2z = x + y$. This means $\pm 3 = x + y + z = 3z$ and so $z = \pm 1$.

The third equation now tells us that either x or $y = 0$, and so the only candidates for (x, y, z) are $(2, 0, 1)$, $(0, 2, 1)$, $(-2, 0, -1)$ and $(0, -2, -1)$.

It is easy, but crucial, to check that these all work in the original equations.

ALTERNATIVE

We can avoid using the fact that $(x + y + z)^2 = 9$.

We may substitute $2z = x + y$ into the original three equations to obtain

$$x^2 + y(x + y) = 4, \quad y^2 + x(x + y) = 4, \quad (x + y)^2 + 8xy = 4$$

The first two are identical, but we may subtract them from the third to obtain $9xy = 0$. So $x = 0$ or $y = 0$. Now $z^2 + 2xy = 1$ gives $z = \pm 1$, so we may calculate the same solutions as above.

ALTERNATIVE

A sophisticated approach exploiting the algebraic symmetry of the equations uses complex numbers. We give a sketch of the argument below, omitting much of the detail.

Let $\omega = \frac{-1+\sqrt{3}i}{2}$ be a complex cube root of unity. This number satisfies $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$. If we label the equations given in the question ①, ② and ③ respectively, it is possible to obtain the following equations:

$$
\begin{aligned}
(x + y + z)^2 &= 4 + 4 + 1 = 9 && ① + ② + ③ \\
(x + \omega^2 y + \omega z)^2 &= 4 + 4\omega + \omega^2 = -3\omega^2 && ① + \omega② + \omega^2③ \\
(x + \omega y + \omega^2 z)^2 &= 4 + 4\omega^2 + \omega = -3\omega^4 && ① + \omega^2② + \omega③
\end{aligned}
$$

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Taking square roots and adding yields

$$3x = \pm 3 \pm \sqrt{3}i(\omega^2 \pm \omega).$$

The three $\pm$ signs are independent, giving eight possible values of x (some are repeated). Since $(\omega^2 - \omega)^2 = -3$, the four combinations where we choose a minus sign in the bracket give real solutions. These correspond to those found earlier and can be used to extract the values of y and z.

Remark

It is possible (but not easy!) to eliminate x and y directly to obtain this polynomial in z:

$$(z-1)(z+1)(3z^2 - 6z + 7)(3z^2 + 6z + 7) = 0$$

It is also possible (but even harder!) to obtain this polynomial in x:

$$x(x-2)(x+2)(3x^2 - 6x + 4)(3x^2 + 6x + 4) = 0$$

All the quadratic factors have negative discriminants, so do not give any real solutions.

Markers' comments

The majority of candidates were successful in finding all four real triples, but they were much less successful in proving that there are no others.

Some candidates assumed that x, y and z have to be integers, which makes the question trivial.

Many candidates found $(x+y)(x-y) = 2z(x-y)$ or a similar equation, then divided by $x - y$ to get $x + y = 2z$. This ignores the possibility that $x - y = 0$. Although this leads only to complex solutions, it is a vital part of the question to show this. So the key to success on this question was to deduce that $x = y$ or $x + y = 2z$ (with variations allowed for the second of these cases). Equivalently, they show that $x - z = \pm(y - z)$.

Many successful candidates incurred a minor penalty for failing to check their solutions in the original equations. This is not just a sensible precaution; it is essential to the logic of non-linear simultaneous equations of this type. We may deduce from the equations that $x = 0$ or $y = 0$ and $z = \pm 1$ are the only possibilities. If we substitute $x = 0$ and $z = 1$ into the equation $y^2 + 2zx = 4$, we will conclude that $y = \pm 2$. Now $(0, 2, 1)$ works in the equation $x^2 + 2yz = 4$, but $(0, -2, 1)$ does not.

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Question 3

An 11×11 square grid is dissected into 11 pieces as follows. At each step, remove the top row and the leftmost column of the remaining grid as a single piece. Thus, the pieces obtained are L-shaped with sizes $21, 19, \ldots, 3$, together with a single square.

The pieces are rearranged to reassemble an 11×11 grid, with rotations allowed (but the pieces may not be turned over) so that the single square occupies the cell in the third row and third column (counting from the top left).

How many ways can this be done?

Solution

There are 2025 ways.

An L-shape can be thought of as an $n \times n$ square with an $n-1 \times n-1$ square removed. We place the L-shapes in order from largest to smallest. At each stage, the unused space is a square, and the L-shape must be placed in a corner of it because of its size. This leaves a smaller square. We keep track of whether the L-shape is placed on the *Left* or the *Right* (L/R) and at the *Top* or the *Bottom* (T/B). In all, to make the single square end up in the third column, we must place two L-shapes on the left and eight on the right. We must also place two L-shapes at the top and eight at the bottom. The choices of (L/R) and (T/B) are independent of each other. Thus the number of ways of reassembling the board is the number of anagrams of 2 L's and 8 R's, multiplied by the number of anagrams of 2 B's and 8 T's. This is

$$\binom{10}{2}^2 = \left(\frac{10!}{2!8!}\right)^2 = \left(\frac{10 \times 9}{2 \times 1}\right)^2 = 2025.$$

Alternative

Rather than dealing with (L/R) and (T/B) independently, we can work with four types of L-shape: W=TL, X=TR, Y=BR, Z=BL. Now we end up with the single square in the third row or column precisely if the tile types (from largest to smallest) form an anagram of one of the following strings:

WWYYYYYYYY, WXZYYYYYYY, XXZZYYYYYY.

Using multinomial coefficients, we can count these as follows:

$$\binom{10}{2,8} + \binom{10}{1,1,1,7} + \binom{10}{2,2,6} = \frac{10!}{2!8!} + \frac{10!}{1!1!1!7!} + \frac{10!}{2!2!6!} = 45 + 720 + 1260 = 2025.$$

Remark

In the argument above it is important that we work 'from the outside in', considering the position of the L-shapes from largest to smallest. This approach makes it clear that the arrangement of shapes is what we will call *nested*, meaning that, for all $k \geq 1$ the k smallest pieces form a square.

If we try to study the arrangement 'from the inside out' it is tempting to assume the arrangement will be nested, so that, for example, the piece of area three will share two edges with the unit

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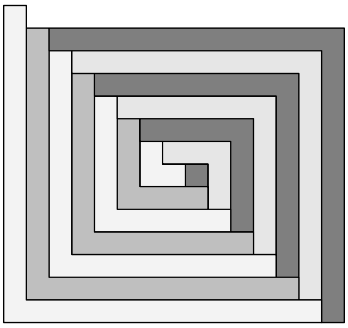
square. However, it is not at all clear how to prove this assertion, leading to a fundamentally flawed argument. It is worth noting that if we consider an infinite version of the problem, namely that of tiling the entire plane with one copy of every possible size of L-shape, we find it can be done without the pieces being nested, as shown below.



Alternative

Let $w(n, r, c)$ be the number of arrangements for a square of side n, where the single square is in row r and column c.

The largest L-shape can go in one of four places due to its size. If we place it in top left (type W above) we can complete the arrangement by filling a square of side $n - 1$ with the single square is in row $r - 1$ and column $c - 1$. These arrangements contribute $w(n - 1, r - 1, c - 1)$ to the value of $w(n, r, c)$. Arguing in an analogous manner when the largest L-shape is type X, Y and Z we obtain the recursive formula:

$$w(n, r, c) = w(n - 1, r - 1, c - 1) + w(n - 1, r - 1, c) + w(n - 1, r, c - 1) + w(n - 1, r, c).$$

Our aim is to find the value of $w(11, 3, 3)$ and we may begin with the boundary case $w(m, 1, 1) = 1$ for all m and the convention that $w(n, r, c) = 0$ if r or c is zero or greater than n.

This allows us to construct a sequence of tables of values of $w(n, r, c)$ for increasing values of n.

$r \backslash c$	1	2	3
1	1	1	0
2	1	1	0
3	0	0	0

$n = 2$

$r \backslash c$	1	2	3
1	1	2	1
2	2	4	2
3	1	2	1

$n = 3$

$r \backslash c$	1	2	3
1	1	3	3
2	3	9	9
3	3	9	9

$n = 4$

$r \backslash c$	1	2	3
1	1	4	6
2	4	16	24
3	6	24	36

$n = 5$

$r \backslash c$	1	2	3
1	1	5	10
2	5	25	50
3	10	50	100

$n = 6$

At this stage there are various patterns we could observe and prove via routine induction arguments. (Here the results on each line depend on those on the line above.)

$w(n, 1, 2) = w(n, 2, 1) = n - 1$

$w(n, 1, 3) = w(n, 3, 1) = \frac{1}{2}(n - 1)(n - 2)$ and $w(n, 2, 2) = (n - 1)^2$

$w(n, 2, 3) = w(n, 3, 2) = \frac{1}{2}(n - 1)^2(n - 2)$

$w(n, 3, 3) = \frac{1}{4}(n - 1)^2(n - 2)^2$

Alternatively, we can simply summarise all the relevant values in a table like the one below.

n	(1,1)	(1,2) (2,1)	(1,3) (3,1)	(2,2)	(2,3) (3,2)	(3,3)
1	1	0	0	0	0	0
2	1	1	0	1	0	0
3	1	2	1	4	2	1
4	1	3	3	9	9	9
5	1	4	6	16	24	36
6	1	5	10	25	50	100
7	1	6	15	36	90	225
8	1	7	21	49	147	441
9	1	8	28	64	224	784
10	1	9	36	81	324	1296
11	1	10	45	100	450	2025

Markers' comments

There are two ways to tackle this problem: a "combinatorial" approach (first two solutions given earlier), and a "recursive" approach (third solution). The candidates scoring highest overall favoured combinatorial approaches. Lower-scoring candidates often attempted an intricate case analysis which generally failed if the recursive nature of those cases was not exploited.

Combinatorial and recursive solutions both rely on the fact that all tilings (irrespective of the final position of the unit square) are *nested*. Candidates who took the recursive route usually addressed nesting to an adequate degree as a byproduct of proving the recurrence. However, a sizeable minority of otherwise-successful candidates who took the combinatorial route failed to address nesting; and since nesting is absolutely crucial to combinatorial approaches, those scripts were heavily penalised. A large number of scripts arriving at the number 2025 scored less than half-marks because of this issue.

Combinatorial approaches boil down to counting a few combinations and permutations. Candidates' success with factorials varied considerably. Markers were forgiving of trivial slips (counting 11 L-tiles instead of 10) and other very small errors. Larger errors typically arose from splitting the problem incorrectly into a large number of overlapping or incomplete sub-cases, and those scripts scored few marks. Candidates who relied silently upon independence of the two $\binom{10}{2}$ in the first solution suffered a small penalty.

Recursive approaches needed to include a reasonably solid proof of the recursion itself, almost always with reference to the effect on the unit square of adding or removing an L-tile. With the recursion established, including base cases, successful candidates usually just stepped out the recursion for positions in the first three rows and columns of the grid, fully solving the problem. Braver candidates attempted to derive $w(n, 3, 3)$ as a polynomial in n, often incurring penalties for algebraic slips. Candidates who used pattern-spotting rather than proof for completing their tables of counts did not score well.

Question 4

Let ABC be an acute-angled triangle with $AB > AC$. Let M be the midpoint of BC. The circle passing through M that is tangent to AB at B and the circle passing through M that is tangent to AC at C intersect again at D.

Prove that $MA \times MD = MB \times MC$.

Solution

We are given Figure 1. The lines AB and AC are tangent to corresponding circles, so the alternate segment theorem is in play.

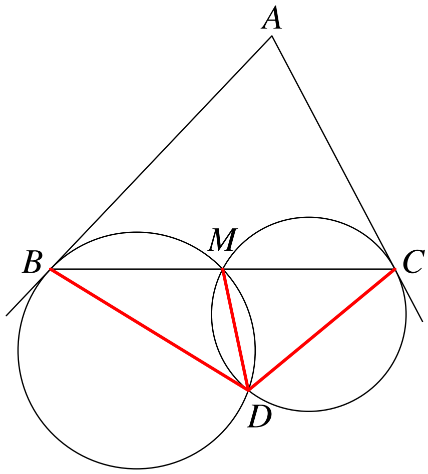


Figure 1

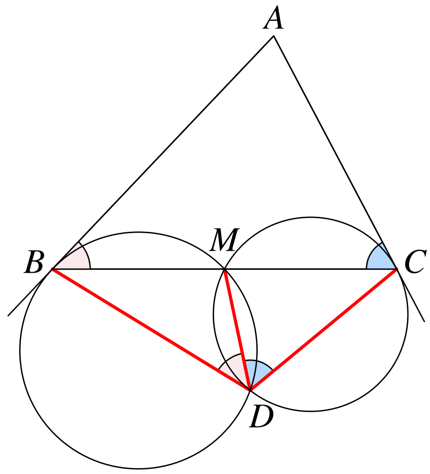


Figure 2

By angle in the alternate segment, we have $\angle B = \angle BDM$ and $\angle C = \angle MDC$. See Figure 2.

Adding we obtain $\angle BDC = \angle B + \angle C$ which is the supplement of $\angle A$, so $ABDC$ is a cyclic quadrilateral. (Two angles are called *supplementary* or the *supplements* of each other if they sum to 180 degrees.) In other words, D lies on the circumcircle Γ of triangle ABC. See Figure 3.

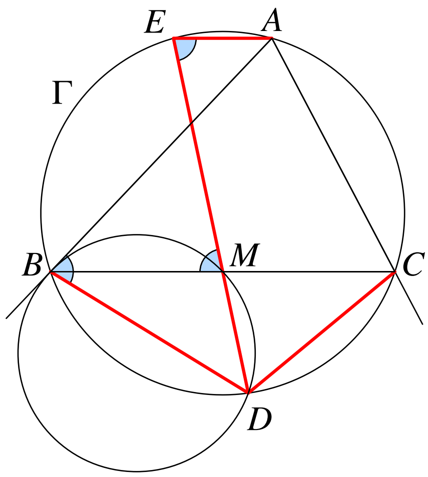


Figure 3

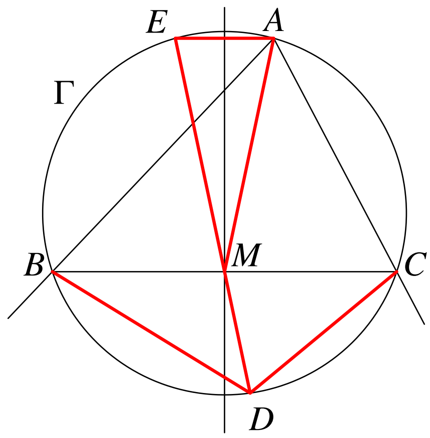


Figure 4

Produce DM to meet the circumcircle Γ again at E. We will prove that EA is parallel to BC. Note that $\angle AED = \angle ABD$ by angle in the same segment. Now using circle BDM and angle in the alternate segment, we have that the supplement of $\angle ABD$ is equal to $\angle DMB$. However, EMD is a straight line so, taking supplements, $\angle ABD = \angle BME$. Now the fact that $\angle BME = \angle AEM$ forces EA and BC to be parallel chords (alternate or Z-angles). See Figure 4.

The perpendicular bisectors of EA and BC are both perpendicular to BC and pass through a common point (the centre of Γ) so they are the same line. Now E is the reflection of A in this line, so $ME = MA$.

Now use the intersecting chord theorem for M with respect to the circle Γ. We have

$$MA \times MD = ME \times MD = MB \times MC.$$

Remark

One could also *define* E to be the reflection of A in the perpendicular bisector of BC and show that $EACDB$ is cyclic.

Remark

An alternative approach along similar lines is to extend line AM to meet the circumcircle again at a point F and show that DF is parallel to BC. This is slightly more involved than the approach using E, but many candidates made it work successfully.

Alternative

As shown in the solution above, we have $\angle BDM = \angle ABC$ and $\angle MDC = \angle BCA$. Rotate triangle BDM clockwise about M until the image D, which we call D', lies on BC. Now rotate triangle CDM anticlockwise until the image of D under this rotation, which we call D'' lies on BC.

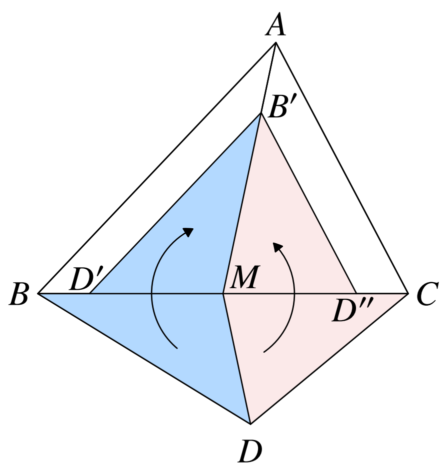


Figure 5

It is easy to check that the images B and C under these respective rotations are the same point, say B'. It is also clear that $B'D'D''$ is similar to ABC. See Figure 5. The segments $B'M$ and AM are corresponding medians of these similar triangles so $B'MD'$ and AMB are similar, giving

$$\frac{B'M}{MD'} = \frac{AM}{MB} \implies B'M \times MB = MD' \times AM$$

Now the facts that $B'M = CM$ and $MD' = MD$ are enough to finish the problem.

REMARK

A closely related variant is to rotate BMD by 180 degrees around M, or, equivalently, to construct D' directly so that $BDCD'$ is a parallelogram without explicitly mentioning the rotation.

REMARK

The line AD is the A-symmedian line of triangle ABC. This means that $\angle CAD = \angle MAB$, so that the lines AM and AD are mutual reflections in either of the angle bisectors (internal or external) of $\angle CAB$. Interested readers may wish to explore the theory of symmedian lines.

MARKERS' COMMENTS

This year's geometry problem was generally found more challenging than recent BMO1 geometry questions. Despite this, there were many fully correct solutions, often written up with clear structure and good use of standard theorems, which the markers were very pleased to see. Some scripts would have benefited from a clear, labeled diagram – an essential feature of any good solution to a geometry problem.

Most successful scripts followed the pattern of the main solution above: first showing that A, B, C, D are concyclic, then introducing a second point on the circumcircle through M and using either parallel chords or symmetry to relate this point to A or D before applying intersecting chords at M.

It was also encouraging to see many candidates gaining partial credit by carrying out this first step – correctly proving that A, B, C, D are concyclic – even when they could not finish the problem. This highlights the importance of writing up any partial progress you make, rather than leaving a question blank.

A key area where candidates lost marks was using an angle equality such as $\angle CBA = \angle MDB$ without referring to the circle theorem (alternate segment) that they were using or relating this to the relevant line being tangent to the circle. There were also scripts that assumed an angle equality such as $\angle MBD = \angle MAB$ without proof which resulted in a heavy penalty.

Several candidates used the "gluing" or reflection approach, rotating or reflecting triangle BDM about M and combining it with triangle CDM to form a new triangle similar to ABC. This can be an elegant solution, but it needs to be written up carefully: one must explain why the reflected or glued figure really is a triangle (i.e. D, M, D' is a straight line), and then clearly indicate which pair(s) of smaller triangles are being used to deduce the final length relation.

Among scripts using the point F on AM, a common gap was to prove an angle equality like $\angle BFM = \angle MDC$ and then immediately assert that F and D are reflections in the perpendicular bisector of BC. This angle equality, even together with $MB = MC$, does not by itself uniquely determine F as the reflection of D. Additional justification is needed (for instance, showing carefully how the properties of the circumcircle and the midpoint condition force F to be the reflected point).

Question 5

George defines a sequence of positive integers $t_1, t_2, t_3, \ldots$ as follows.

He first sets $t_1 = 1$. Then, for $n \geq 1$:

- If t_n is even, then $t_{n+1} = t_n/2$.
- If t_n is odd and greater than 1, then t_{n+1} is $t_n/3$ rounded to the nearest integer.
- If $t_n = 1$, then $t_{n+1} = 2025k$ where k is the number of terms equal to 1 amongst $t_1, t_2, \ldots, t_n$.

Does this sequence contain every positive integer?

SOLUTION

After each occurrence of the number 1, we have a strictly decreasing subsequence starting with $2025k$. This will hit 1 again, so the sequence contains every multiple of 2025. Moreover, if k is odd then we will have the subsequence $2025k, 675k, 225k, 75k, 25k$ so our sequence will contain every odd multiple of 25.

Now suppose we wish to show that some integer x occurs in the sequence. We will consider the possible predecessors of x in the sequence.

$2x$ is always a possible predecessor.

If x is odd, $3x$ is a possible predecessor.

If x is even, $3x \pm 1$ are possible predecessors. (These will be odd, so the rules will ensure we divide by 3 to obtain $x \pm 1/3$ which will round to x.)

Our strategy will therefore be to work backwards through the sequence from x, using these rules, until we reach an odd multiple of 25. This is enough to ensure the sequence contains x.

There are many possible approaches and some care is needed to keep track both of the parity of x and whether or not it is a multiple of 5. Working backwards we can have $x, 2x, 6x \pm 1$. We see that $6x + 1$ and $6x - 1$ are both odd and also that at least one of them is not a multiple of 5. This means that if we can get from any odd non-multiple of 5 to an odd multiple of 25, we can get from x to a term in the sequence for any x.

It is now convenient to consider the numbers modulo 25. We claim that by repeatedly doubling any number not divisible by 5, we can obtain an even number congruent to 8 modulo 25, and from here we can reach an odd multiple of 25 by tripling and adding 1.

The claim follows from the following list of powers of 2 modulo 25 which includes every number that is not a multiple of 5. (We note that if we start with a number that is already 8 mod 25, we can double it 20 times to obtain another number that is 8 modulo 25 which is sure to be even.)

$1, 2, 4, 8, 16, 7, 14, 3, 6, 12, 24, 23, 21, 17, 9, 18, 11, 22, 19, 13, 1, \ldots$

REMARK

The fact that repeated multiplication by 2 produces every possible number modulo 25 (apart from the multiples of 5) is not obvious without some calculation. In particular it is not a

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consequence of 2 and 25 having no common factors. The number 4 has no factor in common with 25, but repeated multiplication by 4 only generates every other number in the list above.

Alternative

An alternate strategy is to show that for any natural number x, one of the numbers $6x+1, 6(6x+1)+1, 6(6(6x+1)+1)+1, \ldots$ will be an (odd) multiple of 25. Indeed, this follows by listing the effect of the map $x \mapsto 6x+1$ on the numbers modulo 25.

$$0, 1, 7, 18, 9, 5, 6, 12, 23, 14, 10, 11, 17, 3, 19, 15, 16, 22, 8, 24, 20, 21, 2, 13, 4, 0, \ldots$$

One can also consider the map $x \mapsto 6x-1$ instead.

Alternative

Another variation on the above strategy is to instead consider the number $3(2^{13}x)+1$. This number is congruent to $x+1$ modulo 25, so after enough iterations we will end up with an (odd) multiple of 25.

Alternative

As before, it is enough to show the sequence contains every odd number. (As then it contains $6x+1, 2x, x$.)

For a given odd integer N suppose the sequence contains the number 2^a3^bN+c where $a, b > c \geq 0$.

The next term in the sequence is of the same form. More precisely, if c is even, the next term is $2^{a'}3^bN+c'$ where $a' = a-1$ and $c' = c/2$. If c is odd the next term is $2^a3^{b'}N+c'$ where $b' = b-1$ and $c' = c/3$ rounded to the nearest integer.

Therefore, at each step the exponents a, b decrease by at most 1, while the remainder c decreases by at least 1 unless it reaches zero. Thus c reaches zero before a or b do. From then on c remains zero, a gradually decreases to zero and then b does at which point we reach N as required.

Taking a, b both equal to 2025 we can then choose $c < 2025$ such that 2^a3^bN+c is a multiple of 2025, which we already know will occur in the sequence.

Remark

This solution shows that the number 2025 could be replaced with any other number and the conclusion would still hold.

Markers' comments

Only a minority of candidates made meaningful progress on this problem. The successful scripts included a variety of creative approaches, though the final alternative listed above was extremely rare.

The most common approach was to show that we can repeatedly double any number not divisible by 5 to obtain an even number congruent to 8 modulo 25. Many candidates claimed that this is a consequence of 2 and 25 sharing no common factors, but as noted in the remarks above, this is not the correct reasoning, and so was not awarded marks.

Some care was required in order to give a fully complete solution, in particular when considering parity. For example, many candidates claimed that if $t_n = 3x + 1$, then we would have $t_{n+1} = x$. However, this is only true if x is even. As a result, many candidates had partial solutions where their methods only worked for certain subsets of the natural numbers.

Candidates often quickly realised that the sequence could be broken up into distinct 'blocks' starting at $2025k$ and ending in 1 for every positive integer k. This observation was worth a mark. At this point it was common to observe that the next few terms in this block removed the factors of 2 and 3, leading many candidates to restrict their attention to the case k is prime. This approach was generally unsuccessful as there is no easy way to deduce that every positive integer is contained in this sequence focusing only on $2025p$ where p is prime.

Another common misconception candidates had was the idea that if this sequence contains repeated terms, then it cannot possibly contain every positive integer.

Question 6

There are 1000 lily pads on a pond arranged in a circle and labelled $1, 2, \ldots, 1000$ in order. (The first four lily pads may also be described using the labels 1001, 1002, 1003 and 1004, respectively.) The first n lily pads are each occupied by a frog with the remaining lily pads not occupied.

Each minute, exactly one of the frogs makes a move. Suppose the frog is on lily pad k. That frog may either:

(i) Swim to lily pad $k + 4$ or $k - 4$, provided that it is not occupied; or

(ii) Jump to lily pad $k + 3$ or $k - 3$ provided that this lily pad is not occupied and the two lily pads jumped over are both occupied. When this happens, the two frogs that were jumped over dive into the pond and don't participate in any further moves.

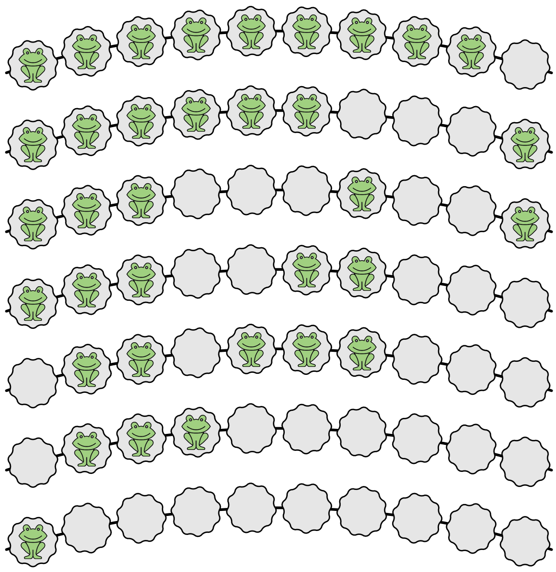
For which values of n is it possible, by a sequence of moves, to end with exactly one frog remaining on the lily pads?

Solution

We begin by observing that neither the jumping move nor the swimming move changes the parity of the number of frogs on the lily pads. This means that if n is even the frogs cannot win, where winning means ending up with a single frog.

If $n = 1$ the frogs have already won, and if $n = 3$ the frogs can win by making a single jump.

Now suppose we have a row of nine occupied lily pads next to a vacant one. The sequence of moves shown below allows us to remove the eight frogs closest to the vacant lily pad.



This means the frogs can win if $n = 8k + 1$ or $n = 8k + 3$ by using this sequence of moves k times. (We note that the required vacant lilypad always exists because 1000 is not of the form $8k + 1$ or $8k + 3$.)

To finish the problem we claim that if $n = 8k + 5$ or $n = 8k + 7$ the frogs cannot win.

We call the lily pads numbered $1, 5, 9, \ldots, 4k + 1$ class 1. We call lily pads $2, 6, 8, \ldots, 4k + 2$ class 2 and define classes 3 and 4 similarly. Now we count the number of frogs in each class.

The swimming move does not change any of these numbers. A jumping move reduces three of these numbers by 1 and increases the remaining class by 1, this means that the parity of the number of frogs in each class changes with every jump.

If we start with $8k + 5$ frogs, the counts in the classes are $4k + 2, 4k + 1, 4k + 1, 4k + 1$, three of which are odd. To reach a state where only one class has an odd number of frogs, the frogs would need to make an odd number, say $2m + 1$, of moves which would leave $8k + 5 - 2(2m + 1) = 4(2k - m) + 3$ frogs. Since $4(2k - m) + 3$ cannot equal 1, the frogs cannot win.

Similarly if we start with $8k+7$ frogs, the counts in the classes are $4k+2, 4k+2, 4k+2, 4k+1$. To reach a state with a single odd class needs an even number, say $2m$, of moves, so the number of frogs left must be $4(2k - m) + 7$ which cannot equal 1.

Remark

The swimming move allows frogs to move freely within their class of lily pad. This observation, together with the notion of an odd class introduced above, allows for a very compact solution.

Suppose $n > 1$ is odd. We claim n frogs can win *if and only if* 4 divides n minus the number of odd classes *and* no class contains more than half the frogs. This can be proved by induction.

Markers' comments

As a hard final question on a hard paper, it was not surprising that this problem did not get very many successful solutions. The question splits into two parts: showing that the possible values of n are possible and showing that the impossible values are impossible. To get full marks, it was necessary to show both. Neither part is easy and many candidates who scored well on one part missed out on credit for the other. Some candidates, having noticed that the three smallest possible values of n are 1, 3 and 9, believed that the answer was powers of 3. While many went on to find constructions showing that powers of 3 are possible, they could not get much credit as their arguments were not easy to adapt to a fully correct solution. Many candidates correctly showed that n must be odd, but this observation alone was not significant enough to be rewarded.

Good attempts often lost marks for their explanation of how the frogs win when they can. Focusing on the number of frogs in each of the four classes of lily pad is an excellent idea for the impossibility part of the proof, but for the construction it introduces some subtleties. For example, it is true that swimming does not change the number of frogs in each class, and it is also true that if we have a frog in each of three classes we can always make a jump 'into' the fourth class. However, the second statement does not follow directly from the first. Instead we must observe that the frogs can swim freely and purposefully within their classes and so arrange themselves in a way that makes the jump possible.

Another issue was candidates who did not check details of the sequence of moves they planned to make, in particular whether it could lead to any class being too full or containing a negative number of frogs at some stage.

Candidates who demonstrated actual frog jumps tended to avoid the two pitfalls mentioned above, but frequently lost marks by giving too little detail as to why a collection of frogs, having jumped over other frogs, are in appropriate positions to be made contiguous by swimming.