British Mathematical Olympiad

Round 2 : Thursday, 15 February 1996

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (28–31 March). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems. The UK Team - six members plus one reserve - for this summer's International Mathematical Olympiad (to be held in New Delhi, India, 7–17 July) will be chosen immediately thereafter. Those selected will be expected to participate in further correspondence work between April and July, and to attend a short residential session 30 June–4 July before leaving for India.

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Determine all sets of non-negative integers x, y and z which satisfy the equation
$$2^x + 3^y = z^2.$$

2. The sides a, b, c and u, v, w of two triangles ABC and UVW are related by the equations
$$u(v + w - u) = a^2,$$
$$v(w + u - v) = b^2,$$
$$w(u + v - w) = c^2.$$
Prove that triangle ABC is acute-angled and express the angles U, V, W in terms of A, B, C.

3. Two circles S_1 and S_2 touch each other externally at K; they also touch a circle S internally at A_1 and A_2 respectively. Let P be one point of intersection of S with the common tangent to S_1 and S_2 at K. The line PA_1 meets S_1 again at B_1, and PA_2 meets S_2 again at B_2. Prove that B_1B_2 is a common tangent to S_1 and S_2.

4. Let a, b, c and d be positive real numbers such that
$$a + b + c + d = 12$$
and
$$abcd = 27 + ab + ac + ad + bc + bd + cd.$$
Find all possible values of a, b, c, d satisfying these equations.