The Actuarial Profession
making financial sense of the future

British Mathematical Olympiad

Round 2 : Tuesday, 26 February 2002

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (4 – 7 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend another meeting in Cambridge. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Glasgow, 22 –31 July) will then be chosen.

Do not turn over until **told to do so.**

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1. The altitude from one of the vertices of an acute-angled triangle ABC meets the opposite side at D. From D perpendiculars DE and DF are drawn to the other two sides. Prove that the length of EF is the same whichever vertex is chosen.

2. A conference hall has a round table wth n chairs. There are n delegates to the conference. The first delegate chooses his or her seat arbitrarily. Thereafter the $(k+1)$ th delegate sits k places to the right of the k th delegate, for $1 \le k \le n-1$. (In particular, the second delegate sits next to the first.) No chair can be occupied by more than one delegate.

   Find the set of values n for which this is possible.

3. Prove that the sequence defined by

   $$y_0 = 1, \qquad y_{n+1} = \frac{1}{2}\left(3y_n + \sqrt{5y_n^2 - 4}\right), \quad (n \ge 0)$$

   consists only of integers.

4. Suppose that $B_1, \ldots, B_N$ are N spheres of unit radius arranged in space so that each sphere touches exactly two others externally. Let P be a point outside all these spheres, and let the N points of contact be $C_1, \ldots, C_N$. The length of the tangent from P to the sphere B_i $(1 \le i \le N)$ is denoted by t_i. Prove the product of the quantities t_i is not more than the product of the distances PC_i.