UKMT

Supported by The Actuarial Profession
making financial sense of the future

British Mathematical Olympiad

Round 2 : Tuesday, 24 February 2004

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (1-5 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Athens, 9-18 July) will then be chosen.

Do not turn over until **told to do so.**

UKMT

Supported by The Actuarial Profession
making financial sense of the future

2004 British Mathematical Olympiad
Round 2

1. Let ABC be an equilateral triangle and D an internal point of the side BC. A circle, tangent to BC at D, cuts AB internally at M and N, and AC internally at P and Q.

   Show that $BD + AM + AN = CD + AP + AQ$.

2. Show that there is an integer n with the following properties:

   (i) the binary expansion of n has precisely 2004 0s and 2004 1s;

   (ii) 2004 divides n.

3. (a) Given real numbers a, b, c, with $a + b + c = 0$, prove that

   $$a^3 + b^3 + c^3 > 0 \quad \text{if and only if} \quad a^5 + b^5 + c^5 > 0.$$

   (b) Given real numbers a, b, c, d, with $a + b + c + d = 0$, prove that

   $$a^3 + b^3 + c^3 + d^3 > 0 \quad \text{if and only if} \quad a^5 + b^5 + c^5 + d^5 > 0.$$

4. The real number x between 0 and 1 has decimal representation

   $$0{\cdot}a_1 a_2 a_3 a_4 \ldots$$

   with the following property: the number of *distinct* blocks of the form

   $$a_k a_{k+1} a_{k+2} \ldots a_{k+2003},$$

   as k ranges through all positive integers, is less than or equal to 2004.

   Prove that x is rational.