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British Mathematical Olympiad

Round 2 : Tuesday, 1 February 2005

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (7-11 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Merida, Mexico, 8 - 19 July) will then be chosen.

Do not turn over until **told to do so.**

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1. The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying
$$\frac{1}{x}+\frac{1}{y}=\frac{1}{N}.$$
Prove that N is a perfect square.

2. In triangle ABC, $\angle BAC = 120°$. Let the angle bisectors of angles A, B and C meet the opposite sides in D, E and F respectively.
Prove that the circle on diameter EF passes through D.

3. Let a, b, c be positive real numbers. Prove that
$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2 \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

4. Let $X = \{A_1, A_2, \ldots, A_n\}$ be a set of distinct 3-element subsets of $\{1, 2, \ldots, 36\}$ such that
i) A_i and A_j have non-empty intersection for every i, j.
ii) The intersection of all the elements of X is the empty set.
Show that $n \leq 100$. How many such sets X are there when $n = 100$?