United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday, 29 January 2009

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (2-6 April). At the training session, students sit a pair of IMO-style papers and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's IMO (to be held in Bremen, Germany 13-22 July) will then be chosen.

Do not turn over until **told to do so.**

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2008/9 British Mathematical Olympiad
Round 2

1. Find all solutions in non-negative integers a, b to $\sqrt{a} + \sqrt{b} = \sqrt{2009}$.

2. Let ABC be an acute-angled triangle with $\angle B = \angle C$. Let the circumcentre be O and the orthocentre be H. Prove that the centre of the circle BOH lies on the line AB. *The circumcentre of a triangle is the centre of its circumcircle. The orthocentre of a triangle is the point where its three altitudes meet.*

3. Find all functions f from the real numbers to the real numbers which satisfy
$$f(x^3) + f(y^3) = (x + y)(f(x^2) + f(y^2) - f(xy))$$
for all real numbers x and y.

4. Given a positive integer n, let $b(n)$ denote the number of positive integers whose binary representations occur as blocks of consecutive integers in the binary expansion of n. For example $b(13) = 6$ because $13 = 1101_2$, which contains as consecutive blocks the binary representations of $13 = 1101_2$, $6 = 110_2$, $5 = 101_2$, $3 = 11_2$, $2 = 10_2$ and $1 = 1_2$.

Show that if $n \leq 2500$, then $b(n) \leq 39$, and determine the values of n for which equality holds.