British Mathematical Olympiad

Round 2 : Thursday, 30 January 2014

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1, 2, 3, 4 in order, and the cover sheet at the front.*
- To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 31 January.

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (3–7 April 2014). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this summer's IMO (to be held in Cape Town, South Africa, 3–13 July 2014) will then be chosen.

Do not turn over until **told to do so.**

United Kingdom Mathematics Trust

2013/14 British Mathematical Olympiad
Round 2

1. Every diagonal of a regular polygon with 2014 sides is coloured in one of n colours. Whenever two diagonals cross in the interior, they are of different colours. What is the minimum value of n for which this is possible?

2. Prove that it is impossible to have a cuboid for which the volume, the surface area and the perimeter are numerically equal. *The perimeter of a cuboid is the sum of the lengths of all its twelve edges.*

3. Let $a_0 = 4$ and define a sequence of terms using the formula $a_n = a_{n-1}^2 - a_{n-1}$ for each positive integer n.

   a) Prove that there are infinitely many prime numbers which are factors of at least one term in the sequence;

   b) Are there infinitely many prime numbers which are factors of no term in the sequence?

4. Let ABC be a triangle and P be a point in its interior. Let AP meet the circumcircle of ABC again at A'. The points B' and C' are similarly defined. Let O_A be the circumcentre of BCP. The circumcentres O_B and O_C are similarly defined. Let $O_A{}'$ be the circumcentre of $B'C'P$. The circumcentres $O_B{}'$ and $O_C{}'$ are similarly defined. Prove that the lines $O_AO_A{}', O_BO_B{}'$ and $O_CO_C{}'$ are concurrent.