United Kingdom
Mathematics Trust

British Mathematical Olympiad
Round 2
Thursday 24 January 2019

Instructions

1. Time allowed: $3\frac{1}{2}$ hours. Each question is worth 10 marks.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.
3. Rough work *should* be handed in, but should be clearly marked.
4. One or two *complete* solutions will gain far more credit than partial attempts at all four problems.
5. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
6. Staple all the pages neatly together in the top *left* hand corner, with questions 1, 2, 3, 4 in order, and the cover sheet at the front.
7. To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 25 January. Candidates sitting the paper in time zones more than 3 hours ahead of GMT must sit the paper on Friday 25 January (as defined locally).
8. In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (2–7 April 2019). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this year's IMO (to be held in Bath, United Kingdom 11–22 July 2019) will then be chosen.
9. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

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1. Let ABC be a triangle. Let L be the line through B perpendicular to AB. The perpendicular from A to BC meets L at the point D. The perpendicular bisector of BC meets L at the point P. Let E be the foot of the perpendicular from D to AC.

Prove that triangle BPE is isosceles.

2. For some integer n, a set of n^2 magical chess pieces arrange themselves on a square $n^2 \times n^2$ chessboard composed of n^4 unit squares. At a signal, the chess pieces all teleport to another square of the chessboard such that the distance between the centres of their old and new squares is n. The chess pieces win if, both before and after the signal, there are no two chess pieces in the same row or column. For which values of n can the chess pieces win?

3. Let p be an odd prime. How many non-empty subsets of

$$\{1, 2, 3, \ldots, p-2, p-1\}$$

have a sum which is divisible by p?

4. Find all functions f from the positive real numbers to the positive real numbers for which $f(x) \leq f(y)$ whenever $x \leq y$ and

$$f(x^4) + f(x^2) + f(x) + f(1) = x^4 + x^2 + x + 1$$

for all $x > 0$.