United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 2

Thursday 27 January 2022

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Instructions

1. Time allowed: $3\frac{1}{2}$ hours. Each question is worth 10 marks.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Work in rough first, and then draft your final version carefully before writing up your best attempt.
3. One or two *complete* solutions will gain far more credit than partial attempts at all four problems.
4. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
5. Start each question on an official answer sheet on which there is a QR code.
6. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. **Do not write your name on any additional sheets.**
7. You should write in blue or black ink, but may use pencil and other colours for diagrams.
8. Write on one side of the paper only. Make sure your writing and diagrams are not too faint.
9. You may hand in rough work where it contains calculations, examples or ideas not present in your final attempt; write 'ROUGH' at the top of each page of rough work.
10. Arrange your answer sheets, including rough work, in question order before they are collected. If you are not submitting work for a particular problem, remove the associated answer sheet.
11. To accommodate candidates in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 28 January. Candidates in time zones more than 3 hours ahead of GMT must sit the paper on Friday 28 January (as defined locally).
12. Around 24 high-scoring students eligible to represent the UK at the International Mathematial Olympiad, will be invited to a training session held in Cambridge shortly before Easter.
13. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

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1. For a given positive integer k, we call an integer n a *k-number* if both of the following conditions are satisfied:

(i) The integer n is the product of two positive integers which differ by k.

(ii) The integer n is k less than a square number.

Find all k such that there are infinitely many k-numbers.

2. Find all functions f from the positive integers to the positive integers such that for all x, y we have:

$$2yf(f(x^2)+x) = f(x+1)f(2xy).$$

3. The cards from n identical decks of cards are put into boxes. Each deck contains 50 cards, labelled from 1 to 50. Each box can contain at most 2022 cards. A pile of boxes is said to be *regular* if that pile contains equal numbers of cards with each label. Show that there exists some N such that, if $n \geq N$, then the boxes can be divided into two non-empty regular piles.

4. Let ABC be an acute angled triangle with circumcircle Γ. Let l_B and l_C be the lines perpendicular to BC which pass through B and C respectively. A point T lies on the minor arc BC. The tangent to Γ at T meets l_B and l_C at P_B and P_C respectively. The line through P_B perpendicular to AC and the line through P_C perpendicular to AB meet at a point Q. Given that Q lies on BC, prove that the line AT passes through Q.

(A minor arc of a circle is the shorter of the two arcs with given endpoints.)