UK Maths Trust

British Mathematical Olympiad

Round 2

Wednesday 22 January 2025

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Instructions

1. Do not turn over until the invigilator tells you to do so.
2. Time allowed: $\mathbf{3\frac{1}{2}}$ **hours**.
3. **Full written solutions – not just answers – are required**, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt.
4. **You may hand in rough work** where it contains calculations, examples or ideas not present in your final attempt; write 'ROUGH' at the top of each page of rough work.
5. **Each question carries 10 marks**. One or two complete solutions will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.
6. **Protractors and calculators are forbidden.** Rulers and compasses are permitted, and indeed are encouraged for drawing accurate diagrams in geometry questions.
7. Start each question on an official answer sheet on which there is a **QR code**.
8. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. **Please do not write your name or initials on any additional sheets**.
9. **Write on one side of the paper only**. Make sure your writing and diagrams are not too faint. (Your work will be scanned for marking.) **Arrange your answer sheets, including rough work, in question order before they are collected**. If you are not submitting work for a particular problem, remove the associated answer sheet.
10. To accommodate candidates in other time zones, please do not discuss any aspect of the paper on the internet until **12 noon GMT on Friday 24th January** when the solutions video will be released at https://bmos.ukmt.org.uk and at ukmt.org.uk/competition-papers and also on *YouTube*. Candidates in time zones more than 5 hours ahead of GMT must sit the paper on Thursday 23rd January 2025 (as defined locally). Do not share the content of the paper (including in email) until the videos have been published.
11. Around 24 high-scoring students eligible to represent the UK at the International Mathematical Olympiad will be invited to a training session held in Cambridge around the Easter holidays.

Enquiries about the British Mathematical Olympiad should be sent to: *challenges@ukmt.org.uk*

1. Prove that if n is a positive integer, then $\frac{1}{n}$ can be written as a finite sum of reciprocals of different triangular numbers.

[A *triangular number* is one of the form $\frac{k(k+1)}{2}$ for some positive integer k.]

2. In an acute-angled triangle ABC with $AB < AC$, the incentre is I and the perpendicular bisector of BC meets BI at P and CI at Q. The circles BIQ and CIP meet again at X. The lines AX and BC meet at D.

Prove that D lies on the circle AQP.

3. An $n \times n$ chessboard consists of n^2 cells which are unit squares. Each cell is coloured black or white so that cells with a common edge are different colours. Isaac muddles up the colouring by repeatedly swapping either two complete columns or two complete rows. Elijah wants to restore the original colouring by repeatedly swapping either two complete columns or two complete rows.

In terms of n, what is the largest number of swaps that Elijah might need?

4. How many different sequences of positive integers satisfy $u_1 = 1$ and

$$u_{n+1} = \frac{(u_n^2 + u_n + 1)^{2025}}{u_{n-1}}$$

for all $n \geq 2$?

