NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test, 1978

May 12th 1978 - $3\frac{1}{2}$ hours

1. A plane convex pentagon ABCDE is said to have the "unit triangle property" if the area of each of the triangles ABC, BCD, CDE, DEA, EAB is unity. Show that all plane convex pentagons with the unit triangle property have the same area and that there is an infinite number of such pentagons no two of which are congruent.

2. Given any integer $m > 1$ prove that there exists an infinity of positive integers n such that the last m decimal digits of 5^n form a sequence in which each digit except the last is of opposite parity to its successor: i.e. if one is odd then the next is even and vice versa.

3. Determine with proof all the roots of the equation

$$\sum_{r=1}^{n} (-1)^{r-1} \frac{x(x-1)\ldots(x-r+1)}{(x+1)(x+2)\ldots(x+r)} = \frac{1}{2}$$

where n is a given positive integer.

4. There are sufficient stocks of four different books for a publisher to give a copy of each to each of his n friends. However he decides to distribute presents to them according to the following rules:

   (i) at least one copy of each of the four books is to be given out;

   (ii) each friend is to have exactly two of the four books (these two not being copies of the same book);

   (iii) the friends who receive any one particular book must form a set different from the set of friends who receive any other particular book.

Show that the number of ways in which the distribution can be made is

$$12(3^{n-1} - 1)(2^{n-2} - 1).$$