NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test, 1979

May 10th 1979 - $3\frac{1}{2}$ hours.

1. a, b, c, d are different positive real numbers. Prove that if at least one of the numbers c and d lies between the numbers a and b, or at least one of the numbers a and b lies between the numbers c and d, then

(*) $$\sqrt{(a + b)(c + d)} \geqq \sqrt{ab} + \sqrt{cd} .$$

Otherwise show that the four numbers can be chosen so that (*) is false.

2. Two equilateral triangles have a common vertex C. Going round each triangle in the anti-clockwise direction the vertices are lettered C, A, B and C, A', B'. O is the centre of triangle CAB and neither A' nor B' coincides with O. M is the midpoint of $A'B$ and N is the midpoint of AB'. Prove that triangles $OB'M$ and $OA'N$ are similar.

3. The sequence of positive integers a_n is defined by

$$a_0 = 1979, \qquad a_{n+1} = [\sqrt{(a_0 + a_1 + \dots + a_n)}] \qquad (n \geqq 0),$$

where $[x]$ denotes the greatest integer not greater than x. (For example $[3\frac{1}{2}] = 3$ and $[5] = 5$.) Determine a_{1979}.

4. Let $b(k)$ denote the number of 1's in the binary expansion of the non-negative integer k. For example $b(13) = 3$ since 13 is 1101 in binary notation. Prove that for all positive integers n,

$$\sum_{k=0}^{2^n-1} (-1)^{b(k)} k^n = (-1)^n \, 2^{\frac{1}{2}n(n-1)} (n!) .$$