NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test

Friday, 23rd March 1984

Time allowed - $3\frac{1}{2}$ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

Candidates are not expected to attempt all five questions.

1. The triangle ABC is right-angled at C. Find all the points D in the plane satisfying the conditions

$$AD \cdot BC = AC \cdot BD = \frac{1}{\sqrt{2}} AB \cdot CD .$$

2. ABCD is a tetrahedron with $DA = DB = DC = d$ and $AB = BC = CA = e$. M and N are the midpoints of AB and CD. A plane π passes through MN and cuts AD and BC at P and Q respectively.

(i) Prove that $AP/AD = BQ/BC$ (= t, say).

(ii) Determine with proof that value of t, expressed in terms of d and e, which minimises the area of the quadrilateral MQNP.

3. Find, with proof, the maximum and minimum values of

$$\cos\alpha + \cos\beta + \cos\gamma ,$$

where $\alpha \geqslant 0$, $\beta \geqslant 0$, $\gamma \geqslant 0$ and

$$\alpha + \beta + \gamma = \frac{4\pi}{3} .$$

4. Let b_n be the number of ways of expressing the positive integer n as a sum of one or more, not necessarily distinct, powers of 2 ; here 1 (= 2^0) is regarded as a power of 2 . Order of the summands is immaterial, so for instance $b_4 = 4$, the expressions in question being

$$1 + 1 + 1 + 1 \, , \quad 1 + 1 + 2 \, , \quad 2 + 2 \, , \quad 4 \; .$$

Call such an expression *full* if it includes at least one summand 2^i for $0 \leq i \leq k$, where 2^k is the largest summand occurring in it. For example the first two of the above expressions for 4 are full, the others are not. Let c_n be the number of full expressions for n . Prove that

$$b_{n+1} = 2c_n$$

for $n \geq 1$.

5. Let p and q be positive integers. Show that there exists an interval I of length $1/q$ and a polynomial P with integer coefficients such that, for all x in I ,

$$\left| P(x) - \frac{p}{q} \right| \leq \frac{1}{q^2} \; .$$