

# FURTHER INTERNATIONAL SELECTION TEST 1991

Thursday 28th February 1991

Time allowed:  $3\frac{1}{2}$  hours

- Start each question on a fresh sheet of paper.
  - Write on one side of the paper only.
  - On every sheet of working write the number of the question in the top left-hand corner and your name and school in the top right-hand corner.
  - Complete the cover sheet provided and attach it to the front of your script, followed by your answers to questions 1, 2, 3, 4 in order.
  - Staple all the pages neatly together in the top left hand corner.
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*A small number of completed solutions is much better than partial attempts at all problems.*

1. In triangle  $ABC$ ,  $B$  is a right-angle and  $\theta$  is the angle between  $AC$  and the median from  $C$  to  $AB$ .

Prove that  $\sin \theta \leq \frac{1}{3}$ .

2. Twelve dwarfs live in a forest. Each has a two-sided cloak which is blue on one side and red on the other. Some dwarfs consistently wear their cloaks red side in and the rest consistently wear their cloaks red side out.

They agree on the following New Year's resolution:

On the  $n$ th day of the New Year, the  $n$ th dwarf modulo 12 will visit each of his friends. If he finds a majority of these friends wearing their cloaks differently from his, he will immediately reverse his own cloak. Otherwise he will continue as before.

Prove that, sooner or later, no further changes will take place.

*(Friendships are mutual and do not change).*

[Turn over

3. Prove that if the perimeter of a triangle with sides  $a, b, c$  is 2 then

$$a^2 + b^2 + c^2 + 2abc < 2.$$

4. Let  $x$  be a positive real number. Prove that at least one of the numbers  
 $x, 2x, 3x, \dots, 20x$   
contains the digit 2 in its decimal expansion.

Let  $N$  be the smallest positive integer such that, for every positive real number  $x$ , at least one of the numbers

$$x, 2x, 3x, \dots, Nx$$

contains the digit 2 in its decimal expansion. Find lower and upper bounds for  $N$  and, if possible, find  $N$  exactly.

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