FIST 2, Bristol, May, 1985

Q1 Let $x_1, x_2, \ldots x_n$ be real numbers such that $0 \leqslant x_i \leqslant 2$ for each i. Prove that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j| \leqslant n^2$$

When does equality hold?

Q2 The in-circle of ΔABC, where $AB > AC$, touches BC at L, and LM is a diameter of the in-circle. AM produced cuts BC at N.

(i) Prove $NL = AB - AC$.

(ii) A circle S of variable radius touches BC at M. The tangents (other than BC) from B and C to S intersect at P. P moves as the radius of S varies. Find the locus of P.

Q3 The sequence u_n is defined for positive integers by

$$u_1 = 1, \quad u_{n+1} = u_n + \{u_n\sqrt{2}\} \quad (n \geqslant 1).$$

Here $\{x\}$ denotes the nearest integer to x, i.e. the integer M such that $x - \frac{1}{2} \leqslant M < x + \frac{1}{2}$.

Determine, with proof, the final (i.e. rightmost) digit of the integer u_{1985}.

Q4 A, B, C, D are points on a sphere of radius 1. Given that $AB.BC.CA.DA.DB.DC = \frac{512}{27}$, prove ABCD a regular tetrahedron.

Q5 Let B_n be the number of ways of partitioning a set with n elements, i.e. expressing it as the union of one or more non-empty subsets, no two of which have a common element. Eg. $B_3=5$, the partitionings of abc being

a,b,c a,bc b,ac c,ab abc

Let C_n be the number of partitionings in which each subset has more than one element, e.g. $C_3=1$. Prove that for $n > 1$

$$C_n = B_{n-1} - B_{n-2} + B_{n-3} - \cdots + (-1)^n B_1.$$

Q6 Solve in non-negative integers x, y, z the equation

$$5^x . 7^y + 4 = 3^z.$$