British Mathematical Olympiad Committee

FINAL SELECTION TEST

Sunday 18th April 1993

Time allowed : $4\frac{1}{2}$ hours

1. Brummie dwarves are famous for their longevity, for their love of cricket (which they start playing as soon as they can walk), and for their fascination with arithmetical accidents. This year our local dwarfish cricket eleven plus scorer all have different integer ages $a_1, a_2, \ldots, a_{12}$; they bat in age order $a_1 < a_2 < \cdots < a_{11}$ (the youngest dwarves are traditionally required to go first), and the scorer, old Methuselah, is the oldest of the lot.
   On Easter Sunday during our first match, old Methuselah observed that, for each positive integer $n \leq 1993$, there exists a subset of the twelve whose ages have sum equal to n, and that the number eleven batsman was as young as he could possibly be for this to happen.
   Find the value of a_{11}.

2. Determine the number of positive integers $n \leq 1000000$ with the property that the binomial coefficient $\binom{2n}{n}$ is not divisible by 3.

3. Six rods A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1 of lengths $(a_1 + a_2)$, $(a_2 + a_3)$, $(a_3 + a_4)$, $(a_4 + a_5)$, $(a_5 + a_6)$, $(a_6 + a_1)$ respectively, where $a_i > 0$ $(i = 1, 2, \ldots, 6)$, are freely jointed together to form a hexagon $A_1A_2A_3A_4A_5A_6$. Prove that the hexagon can be adjusted so that it takes the shape of a convex hexagon circumscribing a circle.
   Prove also that if the real numbers $a_i\,(i = 1, 2, \ldots, 6)$ are permuted amongst themselves in any way, the six new resulting rods have the same property and that whatever permutation is made the inscribed circle always has the same radius. Find the radius explicitly in terms of elementary symmetric functions of $a_1, a_2, \ldots, a_6$.
   Does there exist a convex hexagon, with side lengths 1, 2, 3, 4, 5, 6 in some order, circumscribing a circle?