First Selection Test: Paper 1

Trinity College, Cambridge

31st March 2012

1. Let $ABCD$ be a cyclic quadrilateral so that BC and AD meet at a point P. Consider a point Q, different from B, on the line BP such that $PQ = BP$, and construct the parallelograms $CAQR$ and $DBCS$. Prove that the points C, Q, R, S are concyclic.

2. Determine the maximum number of kings that can be placed on a 12×12 chessboard so that each king threatens exactly one other king. (No two kings may share a cell, and two kings *threaten* each other if they inhabit orthogonally or diagonally adjacent cells).

3. We write $j(n)$ for the number of ones in the number n when it is written in binary notation. Let $k \geq 2$ be a positive integer.

   (a) Show that there is an increasing sequence of odd positive integers $a_1, a_2, \ldots$ such that $j(a_1 a_2 \cdots a_n) = k$ for all n.

   (b) Show that there is some N such that $j(1 \cdot 3 \cdot \cdots \cdot (2n+1)) > k$ for all $n > N$.

Each question is worth seven marks.
Time permitted: 4 hours, 30 minutes.