

# First Selection Test: Paper 1

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1. A *graph* is a finite collection of vertices, some pairs of which are joined by edges (with no loops, or multiple edges between the same pair of vertices). A graph is *planar* if it is drawn in the plane in such a way that edges do not pass over one another. Show that, given a planar graph, a direction can be chosen for each edge such that no vertex has more than three edges coming outwards from it.
2. Determine all injective functions  $f$  from the set of positive integers to itself satisfying the following condition: If  $S$  is a finite set of positive integers such that  $\sum_{s \in S} \frac{1}{s}$  is an integer, then  $\sum_{s \in S} \frac{1}{f(s)}$  is also an integer.
3. Let  $ABC$  be a triangle such that  $AB \neq AC$ , and let  $O$  be its circumcentre. The bisector of  $\angle BAC$  intersects  $BC$  at  $D$ . Let  $E$  be the reflection of  $D$  with respect to the midpoint of  $BC$ . The line through  $D$  perpendicular to  $BC$  intersects the line  $AO$  at  $X$ . The line through  $E$  perpendicular to  $BC$  intersects the line  $AD$  at  $Y$ . Prove that the quadrilateral  $BXCY$  is cyclic.