

FST 1

Trinity, April 5th 2014

1. The set of positive integers is \mathbb{N} . Determine all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f(n) + f(n + 1) = f(n + 2)f(n + 3) - 2010.$$

2. Given $n > 3$, let X be the set $\{1, 2, \dots, 2n\}$. K is a set of pairs (x, y) with x, y distinct elements of X , satisfying the following conditions:
 - If (x, y) is in K , then (y, x) is not in K .
 - Every element x in X participates in at most 19 pairs in K .

Prove that X can be divided into 5 non-empty, disjoint subsets X_1, X_2, X_3, X_4, X_5 such that for each $1 \leq i \leq 5$, the number of pairs (x, y) in K with x, y both in X_i , is at most $3n$.

3. Let ABC be an acute triangle with orthocentre H . The external bisector of angle $\angle CHB$ intersects AB and AC at D and E respectively. The internal bisector of angle $\angle CAB$ intersects the circumcircle of triangle ADE again at K . Show that HK passes through the midpoint of BC .