FST1

28.3.15

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1. N and K are given positive integers. Some number (at least $N+K$) of students needs to be split into groups. Alison splits the students into N non-empty groups. Hannah splits the students into $N+K$ non-empty groups. Let C be the number of students in a strictly smaller group in Hannah's grouping than in Alison's.

   Find the smallest possible value of C.

2. Let $(a_n)_{n\geq 0}$ be a sequence of integers satisfying

$$a_0 = 1,\ a_1 = 3,\ \text{ and } a_{n+2} = 1 + \left\lfloor \frac{a_{n+1}^2}{a_n} \right\rfloor \text{ for all } n \geq 0.$$

   Prove that $a_n a_{n+2} - a_{n+1}^2 = 2^n$ for every $n \geq 0$.

3. Let $\triangle ABC$ be a triangle. Let P_1 and P_2 be points on the side AB such that P_2 lies on the segment BP_1 and $AP_1 = BP_2$. Similarly, let Q_1 and Q_2 be points on the side BC such that Q_2 lies on the segment BQ_1 and $BQ_2 = CQ_1$. The segments P_1Q_2 and P_2Q_1 meet at R, and the circumcircles of $\triangle P_1P_2R$ and $\triangle Q_1Q_2R$ meet again at S, inside triangle $\triangle P_1Q_1R$. Finally, let M be the midpoint of the side AC.

   Prove that the angles $\angle P_1RS$ and $\angle Q_1RM$ are equal.