First Selection Test 2

5-iv-2004

1. Let a_{ij}, $i = 1, 2, 3$; $j = 1, 2, 3$ be real numbers such that a_{ij} is positive for $i = j$ and negative for $i \neq j$. Prove that there exist positive real numbers c_1, c_2, c_3 such that the numbers

$$a_{11}c_1 + a_{12}c_2 + a_{13}c_3, \quad a_{21}c_1 + a_{22}c_2 + a_{23}c_3, \quad a_{31}c_1 + a_{32}c_2 + a_{33}c_3$$

are all negative, all positive or all zero.

2. Let ABC be a triangle and let P be a point in its interior. Denote by D, E and F the feet of the perpendiculars from P to the lines BC, CA and AB respectively. Suppose that

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2.$$

Denote the excentres of triangle ABC by I_A, I_B and I_C in the natural notation. Prove that P is the circumcentre of triangle $I_AI_BI_C$.

3. The sequence $a_0, a_1, a_2, \ldots$ is defined as follows:

$$a_0 = 2, \quad a_{k+1} = 2a_k^2 - 1 \text{ for } k \geq 0.$$

Prove that if an odd prime number p divides a_n, then 2^{n+3} divides $p^2 - 1$.