UK IMO FST2

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1. The circle Γ and the line l do not intersect. Let AB be the diameter of Γ which is perpendicular to l, with B closer to l than is A. An arbitrary point $C \neq A, B$ is chosen on Γ. The line AC intersects l at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect l at F, and let AF intersect Γ at $G \neq A$. Prove that the reflection of G in AB lies on the line CF.

2. Let n and k be positive integers. There are given n circles in the plane. Every pair of them intersect in two distinct points, and all points of intersection are pairwise distinct. Each intersection point must be coloured with one of n distinct colours and each colour must be used at least once. Exactly k distinct colours must occur on each circle. Find all values of $n \geq 2$ and k for which such a colouring is possible.

3. Let N be a positive integer. Two players Alice and Bob take turns to write numbers from the set $\{1, 2, \ldots, N\}$ on a blackboard. Alice begins the game by writing 1 on her first move. If a player has written n on a certain move, the adversary is then allowed to write either $n+1$ or $2n$ (provided the number does not exceed N). The player who writes N wins. We say that N is of type A or type B according to whether Alice or Bob has a winning strategy.

   (a) Determine whether $N = 2004$ is of type A or type B.

   (b) Find the least $N > 2004$ whose type is different from the type of 2004.