

First Selection Test: Paper 2

Trinity College, Cambridge

2nd April 2012

1. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the condition

$$2f(x) = f(x + y) + f(x + 2y)$$

for all real numbers x and all non-negative real numbers y .

2. Let $A_1A_2A_3A_4$ be a noncyclic quadrilateral. Let O_1 and r_1 be the circumcentre, and the circumradius of the triangle $A_2A_3A_4$, respectively. Define O_2, O_3, O_4 and r_2, r_3, r_4 similarly. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

3. Let m be a positive integer and consider a $m \times m$ board consisting of unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the board. When two ants moving in opposite directions meet, they both turn 90° clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving with perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the board, it falls off and does not reappear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the board, or determine that such a moment does not necessarily exist.

*Each question is worth seven marks.
Time allowed: 4 hours, 30 minutes.*