Next Selection Test: 4 hours 30 minutes

Oundle, May 27, 2003

1. Let \( p_1, p_2, \ldots, p_n \) be distinct prime numbers greater than 3. Show that \( 2^{p_1 p_2 \cdots p_n} + 1 \) has at least \( 4^n \) divisors.

2. Let \( ABC \) be a triangle for which there exists an interior point \( F \) such that \( \angle AFB = \angle BFC = \angle CFA \). Let the lines \( BF \) and \( CF \) meet the sides \( AC \) and \( AB \) at \( D \) and \( E \) respectively. Prove that

\[
AB + AC \geq 4DE.
\]

3. Let \( P \) be a cubic polynomial given by \( P(x) = ax^3 + bx^2 + cx + d \), where \( a, b, c, d \) are integers and \( a \neq 0 \). Suppose that \( xP(x) = yP(y) \) for infinitely many pairs \( x, y \) of integers with \( x \neq y \). Prove that the equation \( P(x) = 0 \) has an integer root.