

NST 1, Eton and Oundle, 2009

1. The convex quadrilateral $ABCD$ has the property that $|AB| = |AC| = |BD|$. Let P be the intersection point of its diagonals. Let O and I respectively be the circumcentre and incentre of triangle ABP . Show that if $O \neq I$, then OI and CD are perpendicular.
2. A positive integer is called *monotonic* if when written in base 10, the digits are weakly increasing. Thus 12226778 is monotonic. Note that a positive integer cannot have first digit which is 0. Prove that for every positive integer n , there is an n -digit monotonic number which is a perfect square.
3. Prove that for any positive real numbers a, b, c, x, y and z we have

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \geq \frac{(a+b+c)^3}{3(x+y+z)}.$$