NST 1

30 May 2010

1. Consider a regular 2007-gon. Find the smallest positive integer k having the property that in every set of k vertices, there are four which form a quadrilateral with three edges being edges of the regular polygon.

2. Let b be a positive real number. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$f(x+y) = f(x) \cdot 3^{b^y + f(y) - 1} + b^x (3^{b^y + f(y) - 1} - b^y)$$

   for all $x, y \in \mathbb{R}$.

3. Let $ABCD$ be a cyclic trapezium with $AD \parallel BC$ and $|AD| < |BC|$. The circle is called Γ, and has centre O. Let P be a variable point on the part of the ray BC that is beyond C. It is given that PA is not tangent to Γ *(GCS: I don't see how it could be, but that is what the question says!)*. The circle with diameter PD meets Γ again at E. Let M be the intersection of the lines BC and DE, and N be the second point of intersection of the line PA and Γ.

   Prove that the lines MN pass through a fixed point as P varies.

Each problem is worth 7 points.
Time: 4 hours 30 minutes.