NST 1

Oundle, May 23 2014

1. Prove that in any set of 2000 different real numbers there exists two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$ such that

$$\left| \frac{a-b}{c-d} - 1 \right| < \frac{1}{100000}.$$

2. Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be different points on the line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there is an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC again at R. Prove that $QB = QR$.

3. Players A and B play a painting game on the real line. Players A has a pot of paint containing four units of black ink. The quantity p of this ink can paint black a closed interval of length p. In each round, player A chooses a positive integer m and provides $1/2^m$ units of ink from the pot. Player B then picks an integer k and uses this ink to blacken the closed interval from $k/2^m$ to $(k+1)/2^m$ (and it is possible that some or all of this interval may have been painted before). The goal of player A is to reach a situation where the pot is empty and the interval $[0, 1]$ is not completely blackened.

   Decide whether or not there is a strategy for player A to win in a finite number of moves.